

**Reliability Modeling with Load-Shared Data and  
Product-Ordering Decisions Considering Uncertainty in  
Logistics Operations**

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**Reliability Modeling with Load-Shared Data and  
Product-Ordering Decisions Considering Uncertainty in  
Logistics Operations**

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## SUMMARY

This dissertation consists of two parts with two different topics. In the first part, we investigate “Load-Share Model” for modeling dependency among components in a multi-component system. Systems, where the components share the total applied load, are often referred to as load sharing systems. Such systems can be found in software reliability models and in multivariate failure-time models in biostatistics. When it comes to load-share models, the most interesting component is the underlying principle that dictates how failure rates of surviving components change after some components in the system fail. This kind of principle depends primarily on the reliability application and how the components within the system interact through the reliability structure function. Until now, research involving load-share models have emphasized the characterization of system reliability under a *known* load-share rule. Methods for reliability analysis based on *unknown* load-share rules have not been fully developed. So, in the first part of this dissertation, 1) we model the dependence between system components through a load-share framework, with the load-sharing rule containing unknown parameters and 2) we derive methods for statistical inference on unknown load-share parameters based on maximum likelihood estimation.

In the second half of this thesis, we extend the existing uncertain supply literature to a case where the supply uncertainty dwells in the logistics operations. Of primary interest in this study is to determine the optimal order amount for the retailer given uncertainty in the supply-chain’s logistics network due to unforeseeable disruption or various types of defects (e.g., shipping damage, missing parts and misplaced products). Mixture distribution models characterize problems from solitary failures and contingent events causing network to function ineffectively. The uncertainty in the number of good products successfully reaching the distribution center and retailer poses a challenge in deciding product-order amounts. Because the commonly used ordering plan developed for maximizing expected profits does



not allow retailers to address concerns about contingencies, this research proposes two improved procedures with risk-averse characteristics towards low probability and high impact events.

**Reliability Modeling with Load-Shared Data and  
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**PART I**

**Load-Share Model**

by

**Hyoungtae Kim**

# CHAPTER 1

## INTRODUCTION

### 1.1 *Problem Description*

In most reliability analysis, it is common to assume that components within a system operate independently. But, to derive more dependable results, it is required to develop reliability models which allow dependencies among components.

Consider a system of  $k$  components in parallel, for which component failure rates change only at the failure time of the other components within the system. For example, if the components have identical distributions with initial (constant) failure rate  $\theta$ , then after the first system component fails, the failure rate of the remaining  $k - 1$  components changes to  $\gamma_1\theta$ , for some  $\gamma_1 > 0$ . After the next component failure, the failure rates of the other  $k - 2$  components change to  $\gamma_2\theta$ , and so on.

This is an example of a *load-share* model, where component failure rates depend on the working state of the other components in the system. Early applications of the load-share system models were investigated by researchers in the textile industry (e.g., Daniels (1945)) for studying the reliability of composite materials. Yarns and cables fail after the last fiber (or wire) in the bundle breaks, thus a bundle of fibers can be considered a parallel system subject to a constant tensile load. An individual fiber fails in time with an individual rate that depends on how the unbroken fibers within the bundle share the load of this stress. Depending on the physical properties of the fiber composite, this load-sharing has different meanings in the failure model. Yarn bundles or untwisted cables tend to spread the stress load uniformly after individual failures (i.e., broken fibers). This leads to an *equal load-share (ELS) rule*, which implies the existence of a constant system load that is distributed equally among the working components. For the exponential model, if a constant load is distributed uniformly among the surviving components, then  $\gamma_i = k/(k - i)$ , for  $i = 1, \dots, k - 1$ . It is an

interesting bi-product of the exponential distribution memoryless property that the sample component lifetimes in the equal load-share model equate to an i.i.d. exponential distributed sample.

## 1.2 Literature Review

Load-share models have been studied by Peirce (1926), Daniels (1945), Freund (1961), Rosen (1964), Coleman (1957a,b), Birnbaum and Saunders (1958), and by Harlow and Phoenix (1978, 1982). Even though Peirce introduced this model originally, Daniels (1945) is the first to develop the mathematical theory of the load share model.

Peirce (1926) considered exhaustively the underlying physical considerations and derives useful formulae for the strength of large bundles. The wider significance of the problem was also recognized by Peirce. He pointed out that a study of the strength properties of certain materials must involve considerations fundamentally similar to those arising in the theory of bundles (called by Peirce ‘composite specimens’), since each element of the material may be thought of as made up of sub-elements arranged both in the series and parallel along a particular direction of stress.

Later, Daniels (1945), Coleman (1957a,b), and Rosen (1964) extended Peirce’s research on the strength of large bundles under the *ELS* assumption, where the load is redistributed equally among unbroken fibers. The purpose of their research was to find the relation between the strength of a bundle and the strengths of its constituent threads. In other words, they adopted load-share models to use the available information of the single thread to evaluate the reliability of a large bundle.

Birnbaum and Saunders (1958) adopted the load-share model to derive a more general lifetime distribution of materials. Phoenix (1978) showed that the system failure time is asymptotically normally distributed as the number of components increases. This extended Coleman’s research on the calculation of the asymptotic mean time to failure.

In some complex settings, a bonding matrix joins the individual fibers as a composite material, and an individual fiber failure affects the load of certain surviving fibers (e.g., neighbors) more than others. This characterizes a *local load-share (LLS) rule*, where a

failed component's load is transferred to adjacent components; the proportion of the load the surviving components inherit depends on their distance to the failed component. A more general *monotone load-share (MLS) rule* assumes only that the load on any individual component is nondecreasing as other items fail. Harlow and Phoenix (1982) first adopted this model to consider bundles with fibers in a circular arrangement. Lee, et al. (1995) introduced the loading diagram to explicitly compute the bundle strength survival distribution under this local load-share rule.

This kind of model dependence is not limited to material testing. The load-sharing framework applies to more general problems of detecting members of a finite population. Assume the resources allocated toward finding a finite set of items are defined globally, rather than assigned individually. Then, once items are detected, resources can be redistributed for the purpose of detecting the remaining items and producing a load sharing model. In most cases, the items are identical to the observer, and an *ELS rule* is appropriate for characterizing the system dependence. The reader can be referred to Kvam and Peña (2002) for various application areas of load-share models.

From this framework, potential applications for the load share model extend far beyond the study of textile strength. In software debugging, the detection time for existing bugs in the software can depend on the number of other bugs in the software that have already been found. The discovery of a critical fault in the software may help reveal or conceal other undetected bugs. In manufacturing, as another example, a part can be considered having failed after the failure of the entire set of welded joints that holds the part together. The failure of one or two welded joints can cause the increase of stress on the remaining joints, inducing a load-share model.

### ***1.3 Research Contribution***

Until now, research involving load-share models has emphasized the characterization of system reliability under a *known* load-share rule. Methods for reliability analysis based on *unknown* load-share rules have not been fully developed. In this dissertation, we construct

statistical methods for estimating the load-share rule that dictates the system interdependency as well as for practical tests of the load-share rule. Specifically, we test to see if the load-share parameters are monotonic. The test is based on order restricted inference and utilizes isotonic regression techniques. For simplicity, we limit our discussion to a simple parallel system of identical components, and we focus on a load-share rule where (working) component failure rates change uniformly after each failure within the system, whose magnitude of change is unknown. One of the major findings from this research is that the load-share rule, when it cannot be assumed exactly, severely hinders statistical inference on the system. We found this from the comparison of the asymptotic variances of parameter estimates with or without the load-share rule.

Our elementary research serves as an initial step toward drawing inference on more general load-share rules. Extensions to more general rules and more complicated system configurations can be considered in subsequent research.

## CHAPTER 2

### MODELED DEPENDENCY

In this chapter, we review the original form of dependency models. For the thorough review on dependency models, the reader is referred to Balakrishnan and Basu (1995). Many researchers studied the exponential distribution to explain the lifetime dependency between two components in a system. Among results of these studies, Gumbel's bivariate exponential (BVE) distribution, Freund's simple load-share model, and Marshall-Olkin's BVE distribution are three popular ones. To begin with, we review the simple one-parameter exponential distribution. Let  $X$  follow the univariate exponential distribution with density function

$$f(x) = \lambda \exp(-\lambda x), x \geq 0, \lambda \geq 0,$$

and distribution function

$$F(x) = 1 - \exp(-\lambda x), \quad x \geq 0.$$

Then the reliability function of  $X$  is  $\bar{F}(x) = \exp(-\lambda x)$ . The exponential distribution as defined in equation 2 has a number of properties. Followings are three popular properties of the exponential distribution:

- 1)  $F(x)$  is absolutely continuous
- 2)  $F(x)$  has a constant failure rate function  $\gamma(x)$ , where  $\gamma(x) = f(x)/\bar{F}(x) = \lambda$
- 3)  $F$  has the lack of memory property, which is defined as:  
$$\bar{F}(x+t) = \bar{F}(x)\bar{F}(t) \quad \forall x, t > 0.$$

Exponential distribution is very popular distribution to describe the lifetime of an object in the field of reliability. And it is natural to consider the multivariate extensions of the univariate distribution when it comes to the distribution which is motivated by a physical phenomena in real world. But, unlike the multivariate normal distribution, there is no single

form of the multivariate extension to the exponential distribution. Basu (1988) defined bivariate lack of memory property (BLMP) and bivariate failure rate (BFR) as shown in following equations, to derive a bivariate distribution  $F(x, y)$  with properties similar to those of univariate exponential distribution.

A bivariate distribution  $F(x, y)$  satisfies the BLMP if

$$\bar{F}(xt, yt) = \bar{F}(x, y)\bar{F}(t, t) \quad \forall x, y, t > 0, \quad (1)$$

where  $\bar{F}(x, y) = P(X > x, Y > y)$  is the bivariate survival function.

The BFR of an absolutely continuous bivariate distribution function with density  $f(x, y)$  is given by

$$\gamma(x, y) = f(x, y)/\bar{F}(x, y), \quad (2)$$

where  $\bar{F}(x, y) > 0$ . The marginal distribution should follow univariate exponential distributions. In following sections we introduce three related models.

## 2.1 Gumbel's BVE Distribution

In 1960, Gumbel proposed three absolutely continuous BVE models. It is noteworthy that none of his models are physically motivated. We introduce the joint distribution of  $(X, Y)$  in each model next.

$$\text{(Model 1)} \quad F(x, y) = 1 - e^{-x} - e^{-y} + e^{-x-y-\delta xy}, \quad x, y \geq 0, 0 \leq \delta \leq 1$$

$$\text{(Model 2)} \quad F(x, y) = (1 - e^{-x})(1 - e^{-y})[1 + \alpha e^{-x-y}], \quad x, y \geq 0, -1 \leq \alpha \leq 1$$

$$\text{(Model 3)} \quad F(x, y) = 1 - e^{-x} - e^{-y} + e^{-(x^\beta + y^\beta)^{1/\beta}}, \quad x, y \geq 0, 1 \leq \beta$$

In each model,  $X$  and  $Y$  are independent whenever  $\delta = 0$ ,  $\alpha = 0$  and  $\beta = 1$ , respectively. Note that, in general the marginal distributions of  $X$  and  $Y$  are not exponential distributions in all three models.

## 2.2 Freund's Load-Share Model

In 1960, Freund developed a model that is physically motivated. He was one of the first who introduced a physically motivated model. Let  $X$  and  $Y$  represent the lifetimes of components  $C_1$  and  $C_2$  of a two-component system. Further, assume  $X \sim \exp(\lambda_1)$  and



$Y \sim \exp(\lambda_2)$ . According to the Freund's model, the failure rate of  $C_2$  changes to  $\lambda'_2$  from  $\lambda_2$ , ( $\lambda'_2 > \lambda_2$ ), upon the failure event of component  $C_1$  because of extra stress. Similarly,  $\lambda_1$  changes to  $\lambda'_1$  in case component  $C_2$  fails first, ( $\lambda'_1 > \lambda_1$ ), due to the same reason. Assuming that  $\lambda_1 + \lambda_2 - \lambda'_2 \neq 0$ , the joint density of  $(X, Y)$  is

$$f(x, y) = \begin{cases} \lambda_1 \lambda'_2 \exp[-\lambda'_2 y - (\lambda_1 + \lambda_2 - \lambda'_2)x], & \text{if } 0 < x < y; \\ \lambda'_1 \lambda_2 \exp[-\lambda'_1 x - (\lambda_1 + \lambda_2 - \lambda'_1)y], & \text{if } 0 < y < x. \end{cases} \quad (3)$$

But, we should note that the marginal distributions of  $X$  and  $Y$  are not exponential distributions in general. These marginal distributions can be shown to be mixtures or weighted averages of exponential distributions as following:

$$f_1(x) = \frac{(\lambda_1 - \lambda'_1)(\lambda_1 + \lambda_2)e^{-(\lambda_1 + \lambda_2)x}}{\lambda_1 + \lambda_2 - \lambda'_1} + \frac{\lambda'_1 \lambda_2 e^{-\lambda'_1 x}}{\lambda_1 + \lambda_2 - \lambda'_1} \quad (4)$$

and

$$f_2(y) = \frac{(\lambda_2 - \lambda'_2)(\lambda_1 + \lambda_2)e^{-(\lambda_1 + \lambda_2)y}}{\lambda_1 + \lambda_2 - \lambda'_2} + \frac{\lambda_1 \lambda'_2 e^{-\lambda'_2 y}}{\lambda_1 + \lambda_2 - \lambda'_2} \quad (5)$$

provided  $\lambda_1 + \lambda_2 \neq \lambda'_1$  and  $\lambda_1 + \lambda_2 \neq \lambda'_2$ , respectively. In this model,  $X$  and  $Y$  are independent if and only if  $\lambda_1 = \lambda'_1$  and  $\lambda_2 = \lambda'_2$ . And if  $\lambda_1 > \lambda'_1$  (or if  $\lambda_2 > \lambda'_2$ ), the expected life of component  $C_1(C_2)$  improves when the other component fails. Software debugging problem is a good example to illustrate this special situation. On the other hand, if  $\lambda_1 < \lambda'_1$  (or if  $\lambda_2 < \lambda'_2$ ), the expected lifetime of  $C_1(C_2)$  decreases upon the failure of component  $C_2(C_1)$ .

### 2.3 Marshall-Olkin's BVE Distribution

The joint survival function of the Marshall-Olkin's distribution (1967) is

$$\bar{F}(x, y) = P(X > x, Y > y) = \exp(-\lambda_1 x - \lambda_2 y - \lambda_3(x, y)), \quad (6)$$

$$x > 0, y > 0, \lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0. \quad (7)$$

This model is known as the bivariate exponential (BVE) distribution because the marginal distributions are univariate exponential distributions.  $X$  and  $Y$  are independent when  $\lambda_3 = 0$ . The BVE distributions can be derived in numerous ways. For example, assuming

a system with two components,  $C_1$  and  $C_2$ , using three independent Poisson processes  $P_1$ ,  $P_2$ , and  $P_3$  with parameters  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda_3$  the BVE distribution can be constructed. Suppose that the failure event of component  $C_1$  follows the first poisson process, the failure event of  $C_2$  occurs according to the second process, and the simultaneous failures of both components happen in accordance with the third poisson process. If we let  $U$ ,  $V$ , and  $W$  be the time of the first event in poisson processes  $P_1$ ,  $P_2$  and  $P_3$ , and let  $X$  be the time until component A's failure and  $Y$  be failure time of  $C_2$ . Then,  $X = \min(U, V)$ ,  $Y = \min(V, W)$ , and  $\bar{F}(x, y) = P(X > x, Y > y) = P(U > x, V > y, W > \max(x, y))$ , which is given by equation (6).

The BVE distribution has many important properties. It satisfies the bivariate lack of memory property (BLMP) as in (1). And it is known as the only solution of the functional equation in (1), subject to the requirement of exponential marginal distributions. But, BVE distribution has both an absolutely continuous and a singular part, which can be expressed as

$$\bar{F}(x, y) = \frac{\lambda_1 + \lambda_2}{\lambda} \bar{F}_a(x, y) + \frac{\lambda_3}{\lambda} \bar{F}_s(x, y), \quad (8)$$

where  $\lambda = \lambda_1 + \lambda_2 + \lambda_3$ ,

$$\bar{F}_s(x, y) = \exp[-\lambda_3 \max(x, y)] \quad (9)$$

is a singular distribution, and

$$\bar{F}_a(x, y) = \frac{\lambda}{\lambda_1 + \lambda_2} \exp[-\lambda_1 x - \lambda_2 y - \lambda_3 \max(x, y)] - \frac{\lambda_3}{\lambda_1 + \lambda_2} \exp[-\lambda_3 \max(x, y)] \quad (10)$$

is absolutely continuous. Here it is interesting to note that  $\min(X, Y)$  is exponentially distributed with parameter  $\lambda$ . This result is similar to the case where  $X$  and  $Y$  are independent random variables.

## 2.4 Discussion

Even though only three models are explained in this chapter, there are numerous other models for the BVE distributions. For the exclusive survey of these models, readers can refer

Basu (1988) and references therein. Unfortunately, there exist few options to engineers for modeling dependent systems. Existing methods for such systems studied in engineering and physical sciences are typically based on two classes of models: shock models and load-share models. Shock models, such as Marshall-Olkin Model (1967) in Section 2.3, enable the user to model component dependencies by incorporating latent variables to allow simultaneous component failures. Even though shock model provides an easier avenue for multivariate modeling of system component lifetimes, dynamic models such as load-share model are more realistic in environments where a component's failure rate can change upon another component's failure in the same system. Freund's model can be viewed as a simple load-share model for a system with two components. But in his model, he did not consider the underlying load-share rules which dictate how failure rates change after some components in the system fail. Weier (1981) is the first one who actually analyzed the reparametrization of the Freund's model. He modeled the post failure hazard rate  $\theta_2$  by  $\gamma\theta_1$ ,  $\gamma > 0$ . Here  $\gamma = 1$  implies independence,  $\gamma > 1$  corresponds to an increased work load on the remaining component, while  $\gamma < 1$  corresponds to a reduced work load. In this dissertation, we extend the Freund's model to the general  $k$  components case and introduce load-share parameters using a reparametrization of the extended Freund's model as in Weier (1981) to make general statistical inferences on the details of possible dependencies among components in a system.

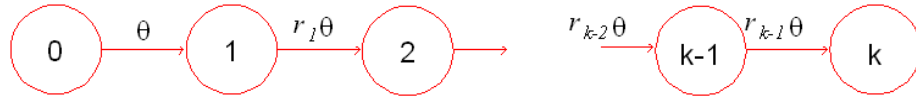
## CHAPTER 3

### STATISTICAL INFERENCE

The reader can be referred to Balakrishnan and Basu (1995) for a survey of techniques for statistical inference about parameter values for various bivariate dependency models, e.g. Gumbel's model, Marshall-Olkin model, Freund's model, etc. Multivariate versions of these models have been developed and available in the literature together with appropriate statistical inference techniques, such as tests for independence among components' lifetime. But, it has not been considered to investigate the characteristic of underlying rule of dependencies in detail. Our study makes it possible to test not only for independence but also for more general dependence relationship among components, such as monotone load-share. In following sections, we consider the inference methods for the general form of a load-share model, which is an extension of Freund's simple load share model.

#### *3.1 Inference for the Load-Share Rule*

We assume the component lifetimes for  $n$  parallel systems are observable, and the individual component failure rates are constant and identical. Upon the first failures of the system, the initial (nominal) failure rate  $\theta$  of the surviving components changes to  $\gamma_1\theta, \gamma_2\theta, \dots, \gamma_{k-1}\theta$  after  $1^{st}$  failure,  $2^{nd}$  failure,  $\dots$ , and  $(k-1)^{th}$  failures, respectively. Figure 1 provides graphical illustration of this transition of failure rates.



**Figure 1:** Failure rate transition in k-component load share model

### 3.1.1 Maximum Likelihood Estimation

We seek maximum likelihood estimators (MLEs) of the  $k$  unknown parameters:  $\theta$  and  $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_{k-1})$ . Suppose the random variable  $X_{ij}$  represents the lifetime of the  $j^{th}$  component in the  $i^{th}$  parallel system and that the random spacing  $T_{ij}$  is the time between  $j^{th}$  failure and  $(j-1)^{th}$  failure for the  $i^{th}$  system. Here  $i = 1, \dots, n$  and  $j = 1, \dots, k$ . The likelihood function for the  $i^{th}$  system is

$$L_i(\theta, \gamma | t_{i1}, t_{i2}, \dots, t_{ik}) = k! \theta^k \prod_{j=1}^{k-1} \gamma_j \exp\left(-\theta \sum_{j=1}^k (k-j+1) \gamma_{j-1} t_{ij}\right)$$

where  $\gamma_0 \equiv 1$  and the likelihood function based on  $n$  samples is

$$L(\theta, \gamma | \mathbf{T}) = (k!)^n (\theta)^{nk} \left( \prod_{j=1}^{k-1} \gamma_j \right)^n \exp\left(-\theta \sum_{i=1}^n \sum_{j=1}^k (k-j+1) \gamma_{j-1} t_{ij}\right) \quad (11)$$

where  $\mathbf{T} = \{t_{ij}; 1 \leq i \leq n, 1 \leq j \leq k\}$ ,  $\theta > 0$  and  $\gamma > \mathbf{0}$ . The corresponding  $k$  log-likelihood equations

$$\frac{\partial \log L}{\partial \theta} = \frac{nk}{\theta} - \sum_{i=1}^n \sum_{j=1}^k (k-j+1) \gamma_{j-1} t_{ij} = 0 \quad (12)$$

and

$$\frac{\partial \log L}{\partial \gamma_{j-1}} = \frac{n}{\gamma_{j-1}} - \theta \sum_{i=1}^n (k-j+1) t_{ij} = 0 \quad j = 2, 3, \dots, k \quad (13)$$

provide no general closed form solutions for the MLEs  $(\hat{\theta}, \hat{\gamma})$ . However, from (12) we obtain

$$\theta = \frac{nk}{\sum_{i=1}^n \sum_{j=1}^k (k-j+1) \gamma_{j-1} t_{ij}}, \quad (14)$$

which, on substitution in (13), yields

$$\gamma_{j-1} - \frac{\sum_{i=1}^n \sum_{j=1}^k (k-j+1) \gamma_{j-1} t_{ij}}{k \sum_{i=1}^n (k-j+1) t_{ij}} = 0, \quad j = 2, 3, \dots, k. \quad (15)$$

Any solution  $(\theta, \gamma)$  of these equations in the space  $[0, \infty)^k$  must be in the  $k - 1$  dimensional subspace  $\{(\theta, \gamma) \in [0, \infty)^k | \theta = nk/\Psi(\gamma)\}$  where

$$\Psi(\gamma) = \sum_{i=1}^n \sum_{j=1}^k (k - j + 1) \gamma_{j-1} t_{ij}.$$

Because the  $k \times k$  Hessian matrix  $\{\partial^2 / \partial \gamma_i \partial \gamma_j \log L\}$  is negative definite (see the Appendix for the proof), for any fixed value of  $\theta$ , there exists a vector  $\gamma$  in this subspace that yields a global maximum for (11), viewed as a function of  $\gamma$  alone. The induced profile likelihood function  $L_p$ , displayed below in (16), is obtained by replacing the parameter  $\theta$  with  $nk/\Psi(\gamma)$  in the likelihood function  $L(\theta, \gamma)$ :

$$L_p(\gamma) = k!^n \left( \frac{nk}{\Psi(\gamma)} \right)^{nk} \left( \prod_{j=1}^k \gamma_{j-1} \right)^n \exp(-nk). \quad (16)$$

Any point which maximizes the likelihood function  $L(\theta(\gamma), \gamma)$  must necessarily maximize  $L_p(\gamma)$ . The  $k - 1$  estimating equations corresponding to (16) are

$$\frac{\partial \log L_p}{\partial \gamma_{j-1}} = -\frac{nkT_j}{\Psi(\gamma)} + \frac{n}{\gamma_{j-1}} = 0, \quad j = 2, 3, \dots, k, \quad (17)$$

where  $T_j = \sum_{i=1}^n (k - j + 1) t_{ij}$ . The MLE for  $\theta$  is then deduced from (12). This leads to the following theorem, with the proof listed in Appendix.

**Theorem 1:** Let  $(\hat{\theta}, \hat{\gamma})$  be the maximum likelihood estimator of  $(\theta, \gamma)$  from (14) and (17) in the exponential load-share model. The MLE exists and is unique. Furthermore, as  $n \rightarrow \infty$ , for  $(\theta, \gamma) > \mathbf{0}_k$ , we have that  $\sqrt{n}\{(\hat{\theta}, \hat{\gamma})' - (\theta, \gamma)'\}$  converges to a  $k$ -parameter Gaussian distribution with mean  $\mathbf{0}_k$  and covariance matrix  $\Sigma$ , where

$$\Sigma = \begin{pmatrix} \theta^2 & -\theta\gamma' \\ -\theta\gamma' & D(\gamma^2) + \gamma\gamma' \end{pmatrix}$$

and  $D(\gamma^2)$  is defined as the diagonal matrix with diagonal elements equal to  $(\gamma_1^2, \dots, \gamma_{k-1}^2)$ .

Compared to an ordinary sample of  $n$  i.i.d. exponential random variables, the asymptotic variance for  $\hat{\theta}$  in Theorem 1 is equal to that of the MLE based on the i.i.d. sample that is  $k$  times smaller in size. Clearly, in a parallel system, an unknown load-share condition is detrimental to any analysis of the system's component lifetime distributions. On the other hand, for systems that fail as a series system if not for the ability to transfer load (e.g., mechanical systems), load sharing actually boosts reliability, but the unknown load-share condition still hinders the statistical inference.

There are a variety of iterative methods designed for solving systems of nonlinear equations in (17). The Gauss-Seidel method (see Ortega and Rheinbold (1970)), is especially well suited for the log-likelihood equations in this problem. The Gauss-Seidel iterations solve the  $k - 1$  nonlinear equations

$$Q_{j-1}(\gamma_1, \gamma_2, \dots, \gamma_{k-1}) = \frac{n}{\gamma_{j-1}} - \frac{nk \sum (k-j+1)t_{ij}}{\sum_{p=1}^n \sum_{q=1}^k (k-q+1)\gamma_{q-1}t_{pq}} = 0, \quad j = 2, \dots, k.$$

The MLE can be solved using the following four steps:

- 1 Choose initial solutions  $\gamma_1^{(0)}, \dots, \gamma_{k-2}^{(0)}$  and solve  $Q_{k-1}(\gamma_1^{(0)}, \dots, \gamma_{k-2}^{(0)}, \gamma_{k-1}) = 0$  for  $\gamma_{k-1}$ . Denote the solution as  $\gamma_{k-1}^{(1)}$ .
- 2 Solve  $Q_{k-2}(\gamma_1^{(0)}, \dots, \gamma_{k-3}^{(0)}, \gamma_{k-2}, \gamma_{k-1}^{(1)}) = 0$  for  $\gamma_{k-2}$  denote the solution as  $\gamma_{k-2}^{(1)}$ .
- 3 Continue in this manner, solving for  $\gamma_{j-1}$  by fixing the other variables at their last solution, and finding  $\gamma_{j-1}^{(1)}$  such  $Q_{j-1} = 0$ .

- 4 Repeat these steps in sufficient number of iterations until convergence to  $\hat{\gamma}_1, \hat{\gamma}_2, \dots, \hat{\gamma}_{k-1}$  has been achieved. Then  $\hat{\theta}$  is computed through (14).

### 3.1.2 Simultaneous Confidence Interval for Load-Share Parameters

Confidence statements and hypothesis tests, based on the likelihood ratio, can be constructed for any combination of the failure rate parameter  $\theta$  and load-share parameters  $\gamma$ . The inverse of the observed Fisher information matrix  $\mathbf{I}_o$  provides an estimate of the covariance in the large sample normal distribution of  $\hat{\beta} - \beta$ , where  $\beta \equiv (\theta, \gamma)$  and  $\hat{\beta}$  is the MLE of  $\beta$ . For large samples, the approximate  $(1 - \alpha)$  confidence ellipsoid for  $(\theta, \gamma) \in (0, \infty)^k$  is

$$(\hat{\beta} - \beta)' \mathbf{I}_o^{-1} (\hat{\beta} - \beta) \leq \chi_{k, \alpha}^2, \quad (18)$$

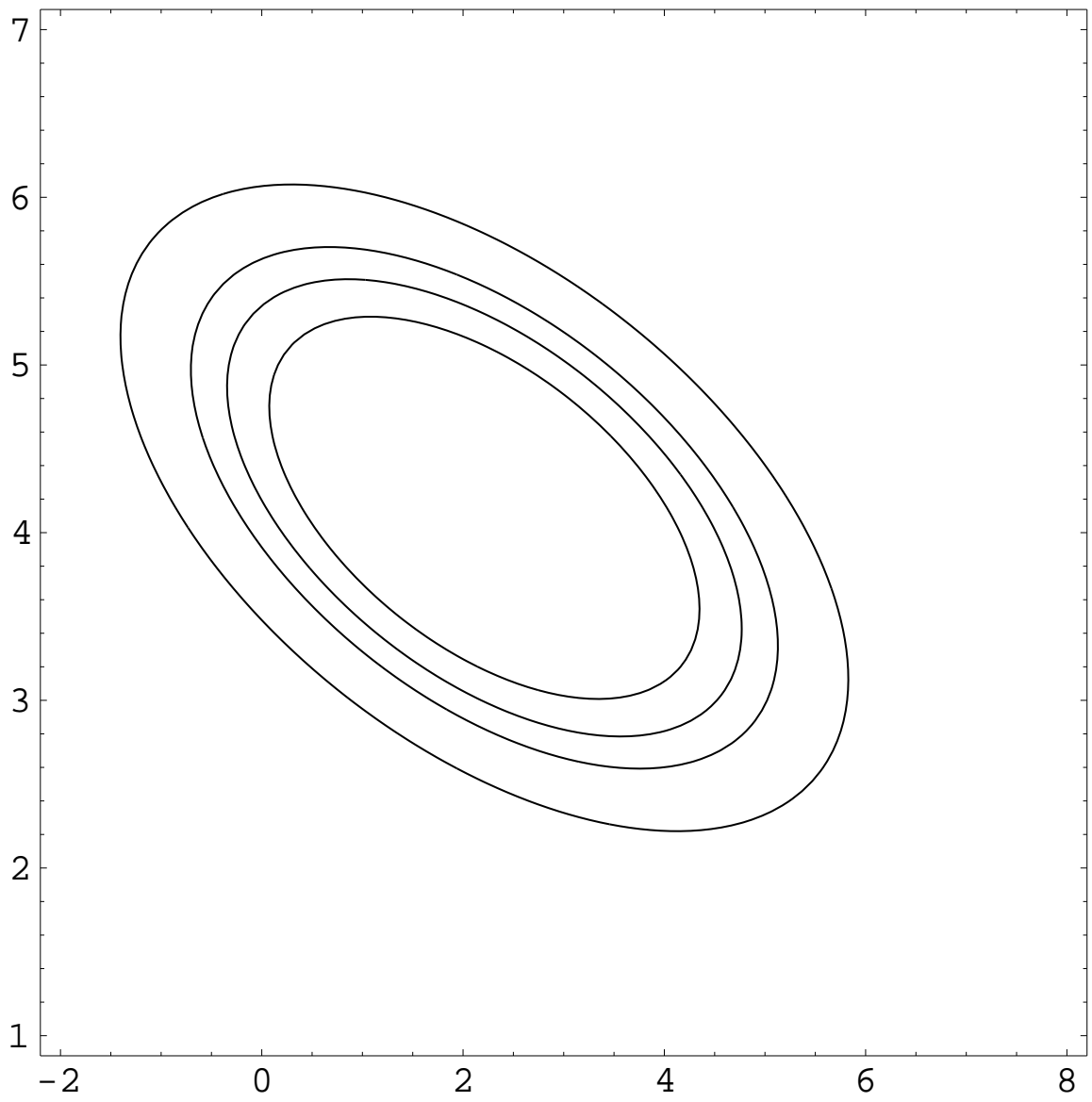
centered at the MLE  $\hat{\beta}$ . Here,  $\chi_{k, \alpha}^2$  is the upper  $\alpha^{th}$  quantile of the chi-square distribution with  $k$  degrees of freedom. The computation of  $\mathbf{I}_o^{-1}$  is included in Appendix.

Consider the following example to illustrate uncertainty estimation for the load-share parameters. Sample data for 10 identical load-share systems were generated by using  $\theta = 0.1$ ,  $\gamma_1 = 2$  and  $\gamma_2 = 4$ . For illustration, we simplify (18) to obtain a confidence region for  $\gamma$  rather than  $\beta$ . The numerical method described above provides MLEs  $\hat{\gamma}_1 = 2.212$ ,  $\hat{\gamma}_2 = 4.148$ , and a  $(1 - \alpha)$  confidence region for  $(\gamma_1, \gamma_2)$ , based on (18) is

$$\frac{1}{n}(\hat{\gamma}_1 - \gamma_1, \hat{\gamma}_2 - \gamma_2) \begin{pmatrix} 2\hat{\gamma}_1^2 & \hat{\gamma}_1\hat{\gamma}_2 \\ \hat{\gamma}_2\hat{\gamma}_1 & 2\hat{\gamma}_2^2 \end{pmatrix} \begin{pmatrix} \hat{\gamma}_1 - \gamma_1 \\ \hat{\gamma}_2 - \gamma_2 \end{pmatrix} \leq \chi_2^2(0.95)$$

or  $81.825 + 0.979\gamma_1^2 - 32.846\gamma_2 + 3.441\gamma_2^2 - 12.389\gamma_1 + 1.943\gamma_1\gamma_2 \leq 5.99$ . Contour lines of the confidence regions for  $(\gamma_1, \gamma_2)$  at  $1 - \alpha = \{0.80, 0.90, 0.95, 0.99\}$  are displayed in Figure 2. The contours convey the strong negative correlation between  $\hat{\gamma}_1$  and  $\hat{\gamma}_2$ .





**Figure 2:** Confidence regions (80%, 90%, 95%, 99%) for  $(\gamma_1, \gamma_2)$

## 3.2 Order Restricted Inference and Monotone Load-Sharing

In many practical applications,  $1 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_{k-1}$  (or *monotone* load-sharing) might be a reasonable assumption; a component failure can cause the increase in the work-load of the other components, which can equate to an increase of failure rate. We consider estimation of the load-share parameters under this order restriction as well as a corresponding test of hypothesis.

### 3.2.1 Order Restricted Maximum Likelihood Estimation

After the  $j^{th}$  failure in the system, the conditional failure rate of the  $k - j$  remaining components is  $\gamma_j \theta$ , so the conditional likelihood between the  $(j - 1)^{th}$  and  $j^{th}$  failure can be computed as  $L_j(\alpha_j) = (k - j + 1)\alpha_j \exp(-(k - j + 1)t_j \alpha_j)$  where  $\alpha_j = \gamma_{j-1}\theta$ ,  $j = 1, \dots, k$ . Then  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$ , where  $\alpha_1 \equiv \theta$ , is isotonic if and only if  $\boldsymbol{\gamma}$  is. The full log-likelihood, in terms of  $\boldsymbol{\alpha}$ , is

$$\log L(\boldsymbol{\alpha}) = n \sum_{j=1}^k \log(k - j + 1) + n \sum_{j=1}^k \log \alpha_j - \sum_{i=1}^n \sum_{j=1}^k (k - j + 1)t_{ij}\alpha_j. \quad (19)$$

The problem of maximizing (19) subject to  $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$  is equivalent to maximizing the log-likelihood

$$\begin{aligned} \log L &= n \sum_{j=1}^k \log \alpha_j - \sum_{i=1}^n \sum_{j=1}^k (k - j + 1)t_{ij}\alpha_j \\ &= \sum_{j=1}^k \left( \frac{n \log \alpha_j}{(k - j + 1) \sum_{i=1}^n t_{ij}} - \alpha_j \right) (k - j + 1) \sum_{i=1}^n t_{ij} \end{aligned}$$

subject to  $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$ .

Wright, Robertson and Dykstra (Chapter 1.5) (1988) showed that the restricted least squares estimate coincides with the maximum likelihood estimate from this log-likelihood

function. For brevity, further references to their book will be denoted by WRD. By applying Theorem 1.4.4 of WRD, the order restricted MLE can be solved as an isotonic regression. Specifically, if we let  $f(j) = \alpha_j$ ,  $g(j) = n / \sum_{i=1}^n t_{ij}(k-j+1)$  and  $w(j) = (k-j+1) \sum_{i=1}^n t_{ij}$ , the solution  $\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_k$  is the isotonic regression  $g^*$  of  $g$  and is given by

$$\begin{aligned} g^*(j) &= \max_{s \leq j} \min_{t \geq j} \frac{\sum_{u=s}^t g(u)w(u)}{\sum_{u=s}^t w(u)} \\ &= \max_{s \leq j} \min_{t \geq j} \frac{n(t-s+1)}{\sum_{u=s}^t (k-u+1) \sum_{i=1}^n t_{iu}}. \end{aligned} \quad (20)$$

The order restricted MLE of  $\gamma_{j-1}$  is  $\tilde{\gamma}_{j-1} = g^*(j)\tilde{\theta}^{-1}$  where  $\tilde{\theta} = g^*(1)$ .

To illustrate the order restricted estimation we generated  $n = 20$  failure times from two systems comprised of three components. The first system is characterized by the parameters  $(\theta = 0.1, \gamma_1 = 1.5, \gamma_2 = 3)$  and the second system by  $(\theta = 0.1, \gamma_1 = 3, \gamma_2 = 1.5)$ . The simulated data are listed in Table 1 and the load-share parameter estimators are listed in Table 2. For System 1, the unrestricted MLEs are already isotonic and thus have the same values as the order-restricted MLEs. For system 2, the corresponding unrestricted MLEs are not isotonic so they do not match the order restricted MLEs.

### 3.2.2 Hypothesis Testing for Unknown Load-Share Rules under Order Restrictions

In this dissertation, three load-sharing rules have been discussed: equal load-sharing, local load-sharing, and monotone load-sharing. Equal load-sharing dictates that at any moment a constant total system load is distributed equally to each working component. As components fail, the total system load remains unchanged, so that the load increases for each of the remaining components according to the rule  $\gamma_i = k/(k-i)$ ,  $i = 1, 2, \dots, k-1$ . As reported in Chapter 1, for the exponential model, this generates another sample of system data that are also i.i.d. exponential. The memoryless property actually preserves the i.i.d. failure data distribution.

A more practical test for reliability applications is for detecting an increasing load within the system:  $H_0 : \gamma_1 = \gamma_2 = \cdots = \gamma_{k-1}$  versus  $H_1 : \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_{k-1}$ . Consistent with the likelihood approach used in estimation, we consider a test based on the likelihood ratio statistic  $(\sup_{H_0} L(\boldsymbol{\beta})) / (\sup_{H_1} L(\boldsymbol{\beta})) = L(\hat{\boldsymbol{\beta}}) / L(\tilde{\boldsymbol{\beta}})$ , where  $\tilde{\boldsymbol{\beta}} = (\tilde{\theta}, \tilde{\gamma})$  are the order restricted MLEs computed in Section 3. In the likelihood function for the monotone load share model, it will be more convenient to work with the notation  $\eta_j = (\theta\gamma_j)^{-1}$  and  $\eta_0 = 1/\theta$ , so an equivalent set of hypotheses is  $H_0 : \eta_0 = \eta_1 = \eta_2 = \cdots = \eta_{k-1}$  versus  $H_1 : \eta_0 \geq \eta_1 \geq \eta_2 \geq \cdots \geq \eta_{k-1}$ .

Here  $H_0$  indicates that there is no actual “load”; the component failure rates remain unchanged after failures within the system. Let  $T_{ij}$  be the time between  $(j-1)^{th}$  failure and  $j^{th}$  failure in  $i^{th}$  sample. Then  $T_{ij}$  is distributed as exponential with failure rate  $(k-j+1)/\eta_{j-1}$ . Under  $H_0$ , the MLEs are  $\hat{\eta}_0 = (\sum_{i=1}^n \sum_{j=1}^k (k-j+1)t_{ij})(nk)^{-1}$  and, for  $j > 1$ ,  $\hat{\eta}_{j-1} = n^{-1} \sum_{i=1}^n (k-j+1)t_{ij}$ . The likelihood ratio statistic is computed by plugging

$$L_0 = \sup_{H_0} L(\boldsymbol{\eta}) = L(\hat{\boldsymbol{\eta}}) = (k!)^n \prod_{j=1}^k \left( \frac{1}{\hat{\eta}_0} \right)^n \exp \left( - \frac{\sum_{i=1}^n \sum_{j=1}^k (k-j+1)t_{ij}}{\hat{\eta}_0} \right)$$

into the numerator. If we define functions  $p_1(\eta_j) = -1/\eta_j$ ,  $p_2(\theta) = 1$ ,  $K(t_j|\theta) = (k-j+1)t_j$ ,  $S(t_j|\theta) = \ln(k-j+1)$  and  $q(\beta_j|\theta) = -\ln \eta_j$ , then regularity conditions 1.5.7, 1.5.8, and 1.5.9 from WRD are satisfied for their Theorem 1.5.2, which proves that under  $H_1$ , the MLE  $\tilde{\boldsymbol{\eta}}$  is solved as the isotonic regression in (20) with weights  $w(x_i) = n$ . For the denominator of the likelihood ratio, we have

$$L_1 = \sup_{H_1} L(\boldsymbol{\eta}) = L(\tilde{\boldsymbol{\eta}}) = (k!)^n \prod_{j=1}^k \left( \frac{1}{\tilde{\eta}_{j-1}} \right)^n \exp \left( - \frac{\sum_{j=1}^k (k-j+1)t_{ij}}{\tilde{\eta}_{j-1}} \right).$$

Due to the order restrictions, we lack the regularity conditions to guarantee the likelihood ratio statistic will have a Chi-square distribution. However, for this particular order restriction, Theorem 4.1.1 of WRD holds and we can approximate the distribution of the test statistic as a mixture of Chi-square distributions. Specifically, if  $T_{01} = -2(\log L_0 - \log L_1)$ , then

$$T_{01} = 2 \sum_{j=1}^k n(-\log \tilde{\eta}_{j-1} + \log \hat{\eta}_0) + 2 \sum_{j=1}^k n\hat{\eta}_{j-1} \left( -\frac{1}{\tilde{\eta}_{j-1}} + \frac{1}{\hat{\eta}_0} \right).$$

Under  $H_0$ , the asymptotic distribution function of  $T_{01}$  is

$$P(T_{01} \leq c) = 1 - \sum_{l=2}^k P(l, k) P(\chi_{l-1}^2 > c). \quad (21)$$

The level probability  $P(l, k)$  denotes the probability that given  $k$  groups under  $H_0$  the isotonic regression will result in  $l$  level sets. Level sets are sets of constancy of isotonic functions, and  $\sum_{l=1}^k P(l, k) = 1$ .

For example, with Sample 1 in Table 1,  $T_{01} = 14.33$  and the P-value  $= P(T_{01} > 14.33) = 0.00023$ , which strongly suggests the ordering described by  $H_1$  is present in the data. For the second sample, we have  $T_{01} = 3.7089$  with P-value  $= 0.053$ . In this case the evidence of load-share parameter ordering is less convincing. For the cases of  $k \in \{3, 4, 5\}$ , Table 3 lists upper quantiles for the null-distribution for this test of hypothesis.

## CHAPTER 4

### CONCLUSION OF PART I

In terms of model uncertainty, there is a slight disadvantage to estimating system or component lifetime distributions in a load-share system when the load-share rule is assumed to be known. An accelerated lifetime model with known acceleration levels is a suitable analog. The inference is more elaborate than inferences for regular lifetime models. However, if the load-share rule cannot be assumed exactly, the load-sharing property severely hinders statistical inference on the system. This was seen in the results of Theorem 1, where it was shown that the variance of the lifetime model parameter estimates was  $k$  times larger than the variances in a regular i.i.d. sample.

This fact can have important ramifications in practical examples in which the failure rates of system components can change after a failure event occurs within the system. If the load-share model is appropriate for a software reliability problem (discussed in Chapter 1), the more traditional modeling and analyses are likely to lead to inference that grossly underestimates parameter uncertainty. For example, in problems where the number of remaining bugs in a piece of software is being estimated, upper bounds for this unknown number of bugs will be too small.

As mentioned in Chapter 1, load-sharing can also serve to benefit system reliability. The ability to transfer a load after a key component failure can save a system that would otherwise fail, such as a system of support structures. The event of the World Trade Center's collapse serves as an example. For primary support, the towers relied on interior columns as

well as pinstripe columns running up each tower's facade, which were turned into additional load-bearing supports. As an afterthought, a system of supports located on the top of each building bound the exterior columns to the core. The structure, called a *hat truss*, was originally installed to hold up antennae, but after the impact of the speeding commercial jet, the hat truss served to spread the load of the damaged columns onto undamaged columns. This load-sharing, as reported in *The New York Times* (see, Glanz and Lipton (2002)), helped prevent the instantaneous collapse of the towers after the planes impact.

This dissertation represents an important first step in drawing inference on load-sharing properties for basic systems. Extending the load-share model to more general lifetime distributions (e.g., Weibull, lognormal, normal) will be problematical in likelihood based inference, undoubtedly. On-going research includes a nonparametric lifetime model under unknown equal load-sharing. In many applications, basic systems of identical components can be modeled adequately by the exponential load-share model if component failure rates remain approximately constant between component failures. In large systems with several components, this is sometimes a common assumption.

**Table 1:** Failure times for load-share samples

n	data 1			data 2		
	$t_{i1}$	$t_{i2}$	$t_{i3}$	$t_{i1}$	$t_{i2}$	$t_{i3}$
1	1.94	0.37	6.93	3.85	6.49	0.36
2	7.44	0.06	2.42	0.32	0.14	7.57
3	0.14	0.20	0.20	8.29	0.12	5.98
4	2.14	1.62	2.34	0.86	6.12	3.43
5	1.91	5.70	1.96	2.42	1.19	6.00
6	8.23	2.25	4.60	1.53	0.20	6.26
7	1.40	2.50	0.09	2.50	1.18	1.01
8	0.79	2.44	7.27	1.30	1.19	9.13
9	0.92	0.12	0.06	4.32	2.08	3.62
10	0.73	0.79	8.61	2.89	0.49	6.28
11	2.78	7.22	1.38	3.25	3.88	6.22
12	0.85	2.81	5.05	17.87	4.18	0.03
13	8.50	4.13	0.52	8.99	0.46	27.63
14	12.93	5.67	1.11	4.08	2.17	15.02
15	4.46	0.96	3.54	1.93	6.81	10.18
16	3.50	7.16	2.38	2.70	0.37	5.04
17	19.59	0.32	1.89	0.34	0.97	2.47
18	4.98	7.32	1.54	5.16	2.64	5.43
19	10.29	2.58	8.61	4.03	0.10	2.38
20	2.22	1.73	1.22	0.16	3.98	2.26

**Table 2:** MLE vs order restricted MLE

	data 1		data 2	
	$\gamma_1$	$\gamma_2$	$\gamma_1$	$\gamma_2$
MLE	1.7875	3.2393	2.2337	1.5837
ORDERED MLE	1.7875	3.2393	1.8534	1.8534

**Table 3:** Upper  $\alpha$ -quantiles for the mixture distribution in (21)

$k$	$\alpha = 0.20$	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$
3	3.047	4.487	5.927	7.363	9.273
4	3.446	4.977	6.491	7.990	9.870
5	3.748	5.345	6.907	8.461	10.486



## CHAPTER 5

### APPENDIX

**Proof of Theorem 1:** First, to show that the existence and uniqueness of the MLE it suffices to prove that matrix  $H$  is negative definite.

The components of the Hessian matrix  $H$  in are given by

$$\begin{aligned} H_{j,j} &= \frac{\partial^2}{\partial \gamma_{j-1}^2} \log L = \frac{nkT_{j+1}^2}{\Psi(\gamma)^2} - \frac{n}{\gamma_j^2}, \quad j = 1, \dots, k-1, \quad \text{and} \\ H_{j,l} &= H_{l,j} = \frac{\partial^2}{\partial \gamma_j \partial \gamma_l} \log L = \frac{nkT_{j+1}T_{l+1}}{\Psi(\gamma)^2}, \quad 1 \leq j \neq l \leq k-1. \end{aligned}$$

Here we establish that  $H$  is negative definite, thus the MLE in (16) exists and is unique.

To show that  $H$  is negative definite, we need  $\mathbf{Z}'\mathbf{H}\mathbf{Z} < 0$ , where  $\mathbf{Z}$  represents a  $k-1$  vector,

$T_j = \sum_{i=1}^n (k-j+1)t_{ij}$ ,  $\Psi(\gamma) = \sum_{j=2}^k \gamma_{j-1}T_j$ , and

$$\mathbf{Z}'\mathbf{H}\mathbf{Z} = \sum_{i=2}^k \sum_{j=2}^k \frac{nkT_iT_jZ_{i-1}Z_{j-1}}{\Psi(\gamma)^2} - n \sum_{j=2}^k \frac{Z_{j-1}^2}{\gamma_{j-1}^2}.$$

We use the following result for the proof:

$$\begin{aligned} \sum \sum_{i \neq j}^k (X_j - X_i)^2 &= (2k-3) \sum_{j=2}^k X_j^2 - 2 \sum \sum_{i \neq j}^k X_i X_j \\ &= (2k-1) \sum_{j=2}^k X_j^2 - 2 \sum_{i=2}^k \sum_{j=2}^k X_i X_j \\ &= 2 \left( k \sum_{j=2}^k X_j^2 - \sum_{i=2}^k \sum_{j=2}^k X_i X_j \right) - \sum_{j=2}^k X_j^2. \end{aligned}$$

If we rearrange the above equation, we have

$$\left( k \sum_{j=2}^k X_j^2 - \sum_{i=2}^k \sum_{j=2}^k X_i X_j \right) = \frac{\sum \sum_{i \neq j}^k (X_j - X_i)^2 + \sum_{j=2}^k X_j^2}{2}.$$

Using the above result together with (17),  $1/\gamma_{j-1} = kT_j/\Psi(\gamma)$ , we can write

$$\begin{aligned}\mathbf{Z}'\mathbf{H}\mathbf{Z} &= \frac{nk}{\Psi(\gamma)^2} \left( \left( \sum_{j=2}^k T_j Z_{j-1} \right)^2 - k \sum_{j=2}^k T_j^2 Z_{j-1}^2 \right) \\ &= -\frac{nk}{2\Psi(\gamma)^2} \left( \sum \sum_{i \neq j}^k (T_j Z_{j-1} - T_i Z_{i-1})^2 + \sum_{j=2}^k T_j^2 Z_{j-1}^2 \right) < 0,\end{aligned}$$

which establishes that  $H$  is negative definite. Therefore the MLE exists and is unique.

For the load-share model, the computation of the covariance, based on the information matrix  $I_\theta = \Sigma^{-1}$  is straightforward. First, the information matrix is easily computed as:

$$I_\theta = \Sigma^{-1} = \begin{pmatrix} \frac{nk}{\theta^2} & \frac{n}{\gamma_1 \theta} & \frac{n}{\gamma_2 \theta} & \cdots & \frac{n}{\gamma_{k-1} \theta} \\ \frac{n}{\gamma_1 \theta} & \frac{n}{\gamma_1^2} & 0 & \cdots & 0 \\ \frac{n}{\gamma_2 \theta} & 0 & \frac{n}{\gamma_2^2} & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ \frac{n}{\gamma_{k-1} \theta} & 0 & 0 & \cdots & \frac{n}{\gamma_{k-1}^2} \end{pmatrix}.$$

Then, it is possible to rewrite it as

$$I_\theta = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$$

where  $I_{11} = nk/\theta^2$ ,  $I_{12} = I_{21}' = n\theta^{-1}\gamma'$ , and  $I_{22} = nD(1/\gamma_1, \dots, 1/\gamma_{k-1})$ . If we define

$$\Sigma = I_\theta^{-1} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

then, from  $I_\theta \times I_\theta^{-1} = \mathbf{I}$ , four equations are obtained for the four unknown submatrices

$\Sigma_{11}, \Sigma_{12}, \Sigma_{21}, \Sigma_{22}$ . They are

$$\Sigma_{11} = (I_{11} - I_{12}\delta^{-1}I_{21})^{-1},$$

$$\Sigma_{12} = -\Sigma_{11}I_{12}I_{22}^{-1},$$

$$\Sigma_{21} = -I_{22}^{-1}I_{21}\Sigma_{11},$$

$$\Sigma_{22} = I_{22}^{-1} - I_{22}^{-1}I_{21}\Sigma_{12}.$$

It is not hard to solve above equations to have  $\Sigma_{11} = n^{-1}\theta^2$ ,  $\Sigma_{12} = \Sigma_{21}' = -n^{-1}\theta\gamma'$ , and

$\Sigma_{22} = n^{-1}(D(\gamma^2 + \gamma\gamma'))$ . Finally, we have the desired covariance matrix:

$$I_{\theta}^{-1} = \Sigma = \frac{1}{n} \begin{pmatrix} \theta^2 & -\gamma_1\theta & -\gamma_2\theta & \cdots & -\gamma_{k-1}\theta \\ -\gamma_1\theta & 2\gamma_1^2 & \gamma_1\gamma_2 & \cdots & \gamma_1\gamma_{k-1} \\ -\gamma_2\theta & \gamma_2\gamma_1 & 2\gamma_2^2 & \cdots & \gamma_2\gamma_{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma_{k-1}\theta & \gamma_{k-1}\gamma_1 & \gamma_{k-1}\gamma_2 & \cdots & 2\gamma_{k-1}^2 \end{pmatrix} = \frac{1}{n} \begin{pmatrix} \theta^2 & -\theta\gamma' \\ -\theta\gamma' & D(\gamma^2) + \gamma\gamma' \end{pmatrix}.$$

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Reliability Modeling with Load-Shared Data and  
Product-Ordering Decisions Considering Uncertainty in  
Logistics Operations

## PART II

# Product-Ordering Decisions Under Logistics Uncertainty

by

Hyoungtae Kim



## CHAPTER 6

### INTRODUCTION

#### ***6.1 Problem Description***

This research seeks to determine the optimal order amount for the retailer given uncertainty in supply-chain's logistics network due to unforeseeable disruption or various types of defects (e.g., shipping damage, missing parts and misplacing products). Mixture distribution models characterize problems from solitary failures and contingent events causing network to function ineffectively. The uncertainty in the number of good products successfully reaching the distribution center and retailer poses a challenge in deciding product-order amounts. Because the commonly used ordering plan developed for maximizing expected profits does not allow retailers to address concerns about contingencies, this research proposes two improved procedures with risk-averse characteristics towards low probability and high impact events. Several examples illustrate the impact of DC's operation policies and model assumptions on retailer's product-ordering plan and resulting sales profit.

#### ***6.2 Motivation and Relevant Literature***

In this era of global sourcing, to reduce purchase costs and attract a larger base of customers, retailers such as Wal-Mart, Home Depot, Dollar General are constantly seeking suppliers with lower prices and finding them at greater and greater distances from their distribution centers (DCs) and stores. In consequence, a significant proportion of shipped products from overseas suppliers is susceptible to defects. Reasons for defects include missing parts, misplaced products (at DCs, stores) or mistakes in orders and shipments. Sometimes, products

are damaged from mishandling in transportation or are affected by the low probability and high impact contingency such as extreme weather, labor dispute and terrorist attack. When there are logistics delays due to security inspections at U.S. borders and seaports or simply by traffic problems, products do not arrive at the DCs or stores on time. Regardless of the problems contributed from supply sources or logistics operations, this research includes them in the *supply and logistics defects*. Two case studies with a major retailing chain indicate that the proportion of the “defects” could reach 20%. This creates significant challenge in product-ordering and shelf-space management.

If the defect rate is not accounted for in the purchase order, the resulting product shortages serve as precursors to several consequences, including inconveniencing customers, compromising the retailer’s reputation for service quality, and then having to trace, sell, repair or return the defective products. Based on our interaction experience with retailers, the stock-out problem can cause more than one billion dollars in a reasonably large size retail chain. On the other hand, use of excessive inventory to handle the uncertain supply and logistics defects will lose a company’s competitive edge. In this dissertation we model the defect process in a three-level supply chain network with many suppliers, one DC and one store, and link this defect model to DCs operation policy for developing an optimal product-ordering scheme.

The literature on supply-chain contract decisions usually utilizes a high-level general model to describe supply uncertainties without getting into any degree of logistics details. See examples in Sculli and Wu (1981), Ramasech et al., (1991), Lau and Lau (1994), Parlar and Perry (1995), Parlar (1997), Weng and McClurg (2003), and Mohebbi (2003) for diverse implications of random production lead-time on inventory policies. Gulyani (2001) studied the effects of poor transportation on the supply chain (i.e., highly ineffective freight

transportation systems) and showed that it increases the probability of incurring damage in transit and total inventories, while also increasing overhead costs. Silver (1976) used the Economic Order Quantity (EOQ) formulation to model the situation that the order quantity received from the supplier does not necessarily match the quantity requisitioned. He showed that the optimal order quantity depends only on the mean and standard deviation of the amount received. Shih (1980) studied the optimal ordering schemes in the case where the proportion of defective products (PDP) in the accepted products has a known probability distribution. Noori and Keller (1984) extended Silver's model to obtain an optimal production quantity under distribution assumptions such as uniform, normal and gamma of the amount of products received at stores.

Literatures concerned with logistics or probabilistic networks are more common in the area of transportation, especially in the hazardous material routing problem. In the transportation of hazardous material, the implication of the accident event of a truck fully loaded with hazardous material can be treated as contingency, but their main objective is to find the optimal route to minimize the expected total system cost. Most of them do not concern the supply chain contract decisions such as the optimal ordering quantity.

In the literature, the supply chain and logistics decision processes are not linked together. Without understanding how logistics works, (e.g., how defects occur in the supply network, whether a different operational policy in the DC does have any effects on the defect process), the supply chain contract decisions might not be accurate, especially in dealing with stochastic optimizations due to supply uncertainties. For example, suppose the typical defect rate is  $\theta_n$  and a contingent event occurs with probability  $p$  (e.g., 0.00001), and upon occurrence,  $\theta_c$  amount of the total shipment is damaged. Then, the overall defect rate is

$(1 - I) \times \theta_n + I \times \theta_c = \theta + I \times (\theta_c - \theta_n)$ , where  $I = 1$  under the contingency and  $I = 0$  otherwise. Even though  $\theta_c$  (e.g., 90%) product damage under the contingent situation can cause enormous stock-out costs to the retailer, the average defect rate  $\theta_n + p \times (\theta_c - \theta_n)$  is nearly the same as  $\theta_n$  without the contingency due to the very small probability  $p$ . Consequently, orders based on the average defect rate (as seen in most of supply-chain contracts) do not prepare the retailer for potentially severe losses that accompany contingencies. Thus, it is important to know how certain defect situations will impact the uncertainty of the amount of good products arrived at stores and develop optimization strategy to encounter these situations.

### **6.3 Contribution**

In this thesis, we suggest two procedures to effectively solve the uncertain supply problem where the uncertainty dwells in the logistics operations. To handle this logistics uncertainty, we map the supply and logistics defects into a model with mixture distributions. Mixture distributions combine two stochastic phenomena according to which the logistics defects arise under normal operational case and under contingency case respectively. Using this model, we demonstrate that conventional risk-neutral solutions do not provide the decision-maker with any protections against a possible contingency. Even when the decision-maker adopts risk-averse solutions of Mean-Variance and Max-Min procedures, the resulting solutions from these methods either fail to reflect the contingency or fail to provide the flexibility. We suggest two procedures that can be used by the decision-maker to generate not only very flexible but also effective solutions under possible contingency.

## 6.4 *Outline*

In this thesis, we describe two procedures with which retailers can generate reasonable solutions that exhibit risk-averse characteristics toward extreme events. To do this, in Chapter 7, we first model processes for product defect rates between any two points in the network, which are directly linked to logistics operations. Next, we construct a random variable representing the total proportion of defective products (TPDP) by integrating models of defects at various stages of the supply-chain network. TPDP is based on a series of mixture distributions and characterizes the overall service levels (defect rates) of logistics operations in contingent and non-contingent circumstances. Moreover, we investigate the impact of two different policies of DCs operations on the resulting distributions of TPDP. Chapter 8 shows the ineffectiveness of using the expected profit in locating the optimal ordering quantity. In Chapter 9, a probability-constrained optimization procedure is developed to handle the low probability and high impact events in logistics uncertainties with numerical examples. Chapter 10 summarizes the results of this dissertation and outlines future research opportunities.

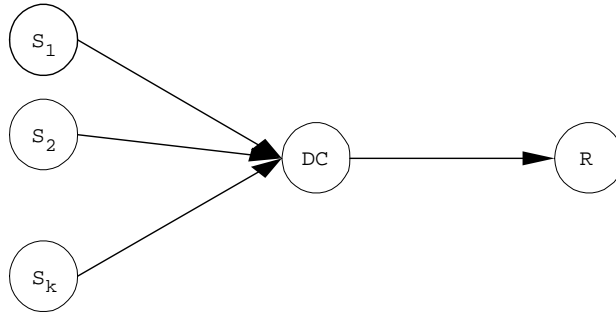
## CHAPTER 7

### UNCERTAIN LOGISTICS OPERATIONS

#### 7.1 *A Damage Model for Logistics Networks*

We consider the problem of a retailer who is buying products from  $k$  identical suppliers. Each supplier provides the retailer with identical products at the same price. Products from the  $k$  suppliers are transported and stored in a single DC before being sent out to the retail outlet (see Figure 3).

A contingent event to products shipped from supplier  $j$  to the DC, denoted by  $X_{jC}$ , is assumed to have high impact, low probability and affects logistics operations (e.g., product damage or delivery delays) between suppliers and the DC. We assume that given a contingency,  $X_{jC}$  is independent of the size of actual shipment, with distribution function  $G_C$ . More generally, we define



**Figure 3:** Supply chain network with  $k$  suppliers and one DC

$X_{jC}$  = PDP due to contingency between supplier  $j$  and DC

$$X_{jC} \sim G_C \text{ where } E(X_{jC}) = \mu_C, \text{Var}(X_{jC}) = \sigma_C^2.$$

$X_{jN}$  = PDP due to non-contingency between supplier  $j$  and DC

$$X_{jN} \sim G_N \text{ where } E(X_{jN}) = \mu_N, \text{Var}(X_{jN}) = \sigma_N^2.$$

$I_j$  = 1 if a contingent event occurs between supplier and DC, = 0 otherwise

and  $\{I_1, \dots, I_k\}$  are independent with  $P(I_j = 1) = p_j$ .

$$P_{jW} = \text{PDP from supplier } j \text{ to DC} = (1 - I_j)X_{jN} + I_jX_{jC}$$

Note that  $P_{jW}$  is a simple mixture distribution, where  $I_j$  serves as the Bernoulli mixing distribution. We consider two scenarios to model how products from the supplier reach the retailer (see Figure 4).

### 7.1.1 Separated Logistics Operations Between Suppliers and Retailer

In the first scenario, illustrated in Figure 4A, different trucks (or other methods of transport) are used for each supplier, so products from different suppliers are not mixed together in transport. We define

$X^*_{jC}$  = supplier  $j$  PDP due to contingency between DC and retailer

$$X^*_{jC} \sim G_C$$

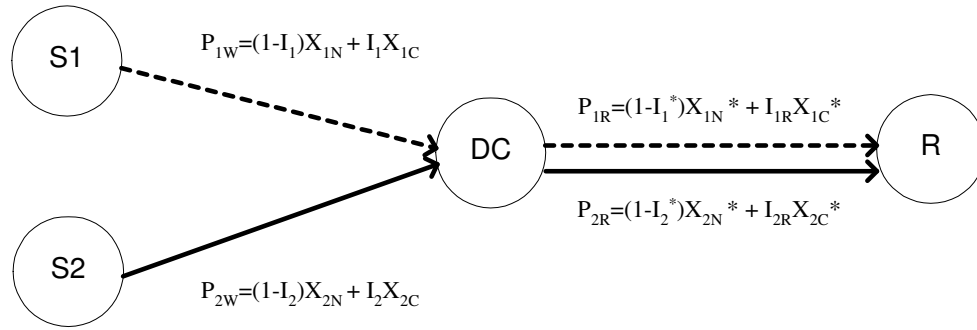
$X^*_{jN}$  = supplier  $j$  PDP due to non-contingency between DC and retailer

$$X^*_{jN} \sim G_N$$

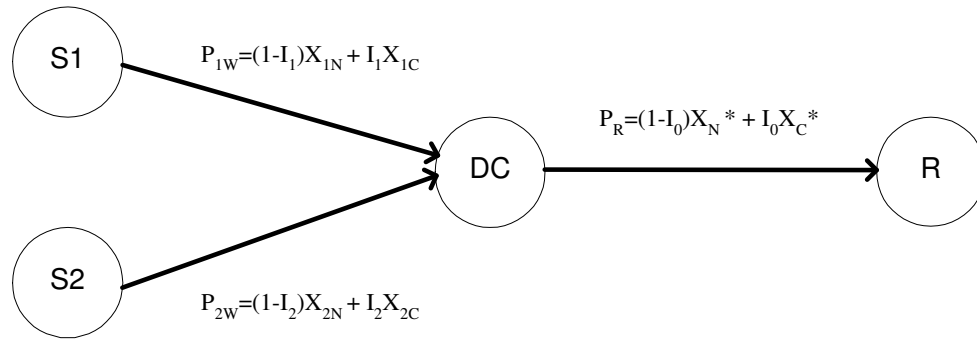
$I^*_j$  = 1 if a contingent event occurs to supplier  $j$  between DC and retailer, = 0 otherwise

and  $\{I^*_1, \dots, I^*_k\}$  are independent with  $P(I^*_j = 1) = p^*_j$

$$P_{jR} = \text{supplier } j \text{ PDP from DC to retailer} = (1 - I^*_j)X^*_{jN} + I^*_jX^*_{jC} \quad j = 1, 2, \dots, k$$



A) Two different trucks from DC to R



B) Same truck from DC to R

**Figure 4:** Different truck-load vs. same truck-load



To derive the distribution of the total proportion of damaged products at the retail level, we first need to derive distributions of the PDP from each supplier that comprises it. For notational convenience, we use random variable  $Y$  to represent the  $TPDP$ . For the non-mixing case we denote this by  $Y_{NM}$ . If we let  $P_j$  be the proportion of damaged products among all the shipment from supplier  $j$ , then we have

$$P_j = P_{jW} + P_{jR}(1 - P_{jW}), \quad \text{and}$$

$$Y_{NM} = \frac{1}{k} \sum_{j=1}^k P_j, \quad j = 1, 2, \dots, k.$$

### 7.1.2 Integrated Logistics Operations Between Suppliers and Retailer

In the “Integrated Operations” scenario, products from different suppliers are delivered from the DC to the retail store using the same transport units (e.g., trucks), and only one indicator variable is required to model the logistics damage under contingency along the route; products loaded in the same truck are exposed to the same risk. Similar to the definitions in Section 7.1.1, let  $P_R$  be the PDP of the remaining undamaged products that are shipped from the DC to the retail store, and define

$$X^*_C = \text{Total PDP due to contingency between DC and retailer}$$

$$X^*_C \sim G_C$$

$$X^*_N = \text{Total PDP due to non-contingency between DC and retailer}$$

$$X^*_N \sim G_N$$

$$I_0 = 1 \text{ if a contingent event occurs between DC and retailer, } = 0 \text{ otherwise}$$

$$\text{with } P(I_0 = 1) = p_0$$

$$P_R = \text{PDP from DC to retailer} = (1 - I_0)X^*_N + I_0X^*_C$$

$$P'_j = \text{Total PDP from supplier } j$$

**Table 4:** Structure of TPDP variables

	Separated ( $P_j$ )		Integrated ( $P'_j$ )	
	normal	contingency	normal	contingency
Supplier $\rightarrow$ DC	$X_{jN}, I_j = 0$	$X_{jC}, I_j = 1$	$X_{jN}, I_j = 0$	$X_{jC}, I_j = 1$
DC $\rightarrow$ Retailer	$X_{jN}^*, I_j^* = 0$	$X_{jC}^*, I_j^* = 1$	$X_N^*, I_0 = 0$	$X_C^*, I_0 = 1$

so for the second scenario,  $Y_M$  represents the aggregate PDP and

$$P'_j = P_{jW} + P_R(1 - P_{jW}),$$

$$Y_M = \frac{1}{k} \sum_{j=1}^k P'_j, j = 1, 2, \dots, k.$$

The total PDPs ( $P'_1, \dots, P'_k$ )s are clearly not independent due to the shared risks. Constructing TPDP variables for the two scenarios is summarized in Table 1.

### 7.1.3 Integrated Supply vs. Separated Supply

The mean and variance of  $Y_{NM}$  can be derived as

$$E[Y_{NM}] = E\left(\frac{\sum_{j=1}^k P_j}{k}\right) = \frac{1}{k} \sum_{j=1}^k E[P_j],$$

$$Var[Y_{NM}] = \frac{\sum_{j=1}^k Var[P_j]}{k^2}.$$

where  $P_j$ s are independent. Similarly, the mean and variance of  $Y_M$  can be derived as

$$E[Y_M] = E\left(\frac{\sum_{j=1}^k P'_j}{k}\right) = \frac{1}{k} \sum_{j=1}^k E[P'_j]$$

$$Var[Y_M] = Var\left(\frac{\sum_{j=1}^k P'_j}{k}\right) = \frac{1}{k^2} \sum_{j=1}^k Var[P'_j] + \frac{1}{k^2} \sum \sum_{i \neq j} Cov(P'_i, P'_j)$$

where

$$\begin{aligned}
Cov[P'_i, P'_j] &= Cov[P_{iW} + P_R(1 - P_{iW}), P_{jW} + P_R(1 - P_{jW})] \\
&= Cov[P_R(1 - P_{iW}), P_R(1 - P_{jW})] \\
&= Var[P_R](1 - E[P_{iW}]) (1 - E[P_{jW}]) \geq 0.
\end{aligned}$$

If we assume  $I_j$ 's,  $I_j^*$ 's, and  $I_0$  to be independent and identically distributed with  $P[I_0 = 1] = p$  then we can further simplify the means and variances for both cases. For the non-mixed case,

$$\begin{aligned}
E[Y_{NM}] &= \frac{1}{k} \sum_{j=1}^k E[P_j] = \frac{1}{k} \sum_{j=1}^k E[P_{jW} + P_{jR}(1 - P_{jW})] \\
&= [(1 - p)\mu_N + p\mu_C] \left( 2 - [(1 - p)\mu_N + p\mu_C] \right), \tag{22}
\end{aligned}$$

$$\begin{aligned}
Var[Y_{NM}] &= \frac{\sum_{j=1}^k Var[P_j]}{k^2} = \frac{\sum_{j=1}^k Var[P_{jW} + P_{jR}(1 - P_{jW})]}{k^2} \\
&= \frac{1}{k^2} \sum_{j=1}^k [(1 - p)\sigma_N^2 + p\sigma_C^2] \left( 2 \left[ 1 - ((1 - p)\mu_N + p\mu_C) \right] + (1 - p)\sigma_N^2 + p\sigma_C^2 \right) \\
&= \frac{1}{k} [(1 - p)\sigma_N^2 + p\sigma_C^2] \left( 2 \left[ 1 - ((1 - p)\mu_N + p\mu_C) \right] + (1 - p)\sigma_N^2 + p\sigma_C^2 \right). \tag{23}
\end{aligned}$$

For the mixed case we have

$$\begin{aligned}
E[Y_M] &= \frac{1}{k} \sum_{j=1}^k E[P'_j] = \frac{1}{k} \sum_{j=1}^k E[P_{jW} + P_R(1 - P_{jW})] \\
&= [(1-p)\mu_N + p\mu_C] \left( 2 - [(1-p)\mu_N + p\mu_C] \right) \tag{24} \\
&= E[Y_{NM}], \\
Var[Y_M] &= Var\left(\frac{\sum_{j=1}^k P'_j}{k}\right) = \frac{1}{k^2} \sum_{j=1}^k Var[P'_j] + \frac{1}{k^2} \sum \sum_{i \neq j} Cov(P'_i, P'_j) \\
&= \frac{1}{k} \left\{ [(1-p)\sigma_N^2 + p\sigma_C^2] \left( 2[1 - ((1-p)\mu_N + p\mu_C)] + (1-p)\sigma_N^2 + p\sigma_C^2 \right) \right\} \\
&\quad + \frac{1}{k^2} \binom{k}{2} \left( (1-p)\sigma_N^2 + p\sigma_C^2 \right) \left( 1 - [(1-p)\mu_N + p\mu_C] \right)^2. \tag{25}
\end{aligned}$$

These results are used in later sections to investigate the effect of contingency on the optimal order quantity and the retailer's profit function.

#### 7.1.4 Risk-Pooling Effects of Integrated Logistics Operations

In this section, we consider an example to illustrate the risk-pooling effects when the decision-maker adopts the mixed supply line strategy. In this example, to focus on the implications of mixed or separate supply lines we disregard all logistics operations between suppliers and DC.

The following notation aids the formulation of the example:

- $Q$  = total order quantity requested by retailer
- $k$  = total number of suppliers, assume  $k = 2$
- $c_1$  = unit wholesale price from supplier 1
- $c_2$  = unit wholesale price from supplier 2, assume  $c_1 < c_2$
- $r$  = unit retail price
- $m_1$  = profit margin of unit product purchased from supplier 1,  $m_1 = r - c_1$
- $m_2$  = profit margin of unit product purchased from supplier 2,  $m_2 = r - c_2$   
(note that  $m_1 > m_2$ )
- $h$  = unit holding cost per period for unsold products
- $\pi$  = unit shortage cost
- $\xi$  = fixed total demand per period
- $P_1$  = *r.v.* representing total proportion of defects among products from supplier 1
- $P_2$  = *r.v.* representing total proportion of defects among products from supplier 2
- $Y$  = *r.v.* representing total proportion of defects among  $Q$  in transit,  $Y = \frac{1}{2}(P_1 + P_2)$ .

Following assumptions are made to show the idea of risk pooling:

- A1) Contingency dictates the magnitude of PDP from each truck
- A2) Under the mixed supply case, number of damages are evenly distributed  
among products from different suppliers
- A3) Under separate supply case, we exclude the situation when  
both trucks experience contingency simultaneously

In addition to these assumptions, we use following discrete distributions for  $P_1$  and  $P_2$

and  $Y$ :

$$P_1 = \begin{cases} 1, & \text{w/p } p; \\ 0, & \text{w/p } 1-p, \end{cases} \quad P_2 = \begin{cases} 1, & \text{w/p } p; \\ 0, & \text{w/p } 1-p, \end{cases} \quad Y = \begin{cases} 0.5, & \text{w/p } p; \\ 0, & \text{w/p } 1-p. \end{cases}$$

We use distributions of  $P_1$  and  $P_2$  for the separate supply case and distribution of  $Y$  for the mixed supply case. According to the assumption A2),  $Y = 0.5$  represents  $(P_1, P_2) = (0.5, 0.5)$ .

Given  $Q$ , the retailer's profit is a function of  $P_1$  and  $P_2$ :

$$\begin{aligned} \Pi(P_1, P_2) &\equiv r \text{Min}[\xi, (1-Y)Q] - c_1(1-P_1)Q/2 - c_2(1-P_2)Q/2 \\ &\quad - h[(1-Y)Q - \xi]^+ - \pi[\xi - (1-Y)Q]^+ \\ &= (r - c_1)(1-P_1)Q/2 + (r - c_2)(1-P_2)Q/2 \\ &\quad - (h+r)[(1-Y)Q - \xi]^+ - \pi[\xi - (1-Y)Q]^+ \\ &= m_1(1-P_1)Q/2 + m_2(1-P_2)Q/2 - (h+r)[(1-Y)Q - \xi]^+ - \pi[\xi - (1-Y)Q]^+ \end{aligned}$$

Under the mixed supply case, the possible profits are:

$$\begin{aligned} \Pi(0, 0) &= (m_1 + m_2)Q/2 - (h+r)(Q - \xi)^+ - \pi(\xi - Q)^+ \\ \Pi(0.5, 0.5) &= (m_1 + m_2)Q/4 - (h+r)(0.5Q - \xi)^+ - \pi(\xi - 0.5Q)^+ \end{aligned}$$

Under the separate supply case, retailer's profit can take one of following forms:

$$\begin{aligned} \Pi(0, 0) &= (m_1 + m_2)Q/2 - (h+r)(Q - \xi)^+ - \pi(\xi - Q)^+ \\ \Pi(1, 0) &= m_2Q/2 - (h+r)(0.5Q - \xi)^+ - \pi(\xi - 0.5Q)^+ \\ \Pi(0, 1) &= m_1Q/2 - (h+r)(0.5Q - \xi)^+ - \pi(\xi - 0.5Q)^+ \\ \Pi(1, 1) &= -\pi\xi \end{aligned}$$

Then the expected profit based on the mixed supply case is:

$$E[\Pi(P_1, P_2)] = p \times \Pi(0.5, 0.5) + (1-p) \times \Pi(0, 0),$$

where the expected profit based on the separated supply case is:

$$\begin{aligned}
E[\Pi(P_1, P_2)] &= p^2 \times \Pi(1, 1) + p(1 - p) \times \Pi(1, 0) \\
&\quad + (1 - p)p \times \Pi(0, 1) + (1 - p)^2 \times \Pi(0, 0) \\
&\simeq p \times \Pi(1, 0) + p \times \Pi(0, 1) + (1 - 2p) \times \Pi(0, 0).
\end{aligned}$$

During the above simplification, we simply dropped all the terms involving  $p^2$ . When there is no contingency (i.e.  $P_1 = 0, P_2 = 0$ ), these two strategies do not cause any differences in profit. But when contingency happens, then the mixed supply case generates equal amount of product damages among mixed products from two suppliers while the supply supply case can cause entire loss of products from each supplier. Because the profit margins of products from two suppliers are different ( $m_1 > m_2$ ), with  $m_2 < (m_1 + m_2)/2 < m_1$  the profit using the separate supply case can be larger and smaller at the same time depending on the source of the damaged products under contingency. But if the decision-maker is risk-averse, the mixed supply line strategy performs better than the separate supply line strategy

## 7.2 *Problem Formulation*

The following notation aids the formulation of the uncertain supply problem in the logistics network. The notation below is little different than the one used in the previous section. We assume same wholesale prices for all the available suppliers and we relaxed the fixed demand assumption:

$Q$	=	total order quantity requested by retailer
$k$	=	total number of suppliers
$c$	=	unit wholesale price (same for all available suppliers)
$r$	=	unit retail price
$h$	=	unit holding cost per period for unsold products
$\pi$	=	unit shortage cost
$\xi$	=	$r.v.$ representing demand per period
$F(\xi)$	=	distribution function of $\xi$ (with p.d.f. $f(x)$ ).
$Y$	=	$r.v.$ representing total proportion damages among $Q$ in transit

In the  $k$ -supplier model, we assume that the total order quantity  $Q$  is split equally between  $k$  suppliers. The retail price is fixed and strictly greater than wholesale cost ( $r > c$ ) regardless of the terms of trade is assumed. The holding cost per period at the retail store level is  $h$  for each unsold product. In the event of a stock out, unmet demand is lost, resulting in the margin being lost (to the retailer). The related stock-out penalty cost is  $\pi$ . All cost parameters are assumed to be known.

The retailer's profit consists of three components: Sales revenue (SR) , procurement costs (PC) from suppliers, and the total system inventory cost (TSIC). After the completion of the logistics operations, the total amount of products received may or may not be enough to meet the demand amount,  $\xi$ . TSIC has two components: overstock inventory cost and total stock-out penalty cost. The shortage amount is primarily due to the unknown actual demand, but partly due to the potential damages to products during the transportation process.

If we use the results from the previous section we can construct the retailer's profit function. For a given  $Q$  and TPDP (either case, mixed or separate supply), the retailer's



profit is

$$\begin{aligned}\Pi(Q, Y) \equiv & r \text{Min}[\xi, (1 - Y)Q] - c(1 - Y)Q \\ & - h[(1 - Y)Q - \xi]^+ - \pi[\xi - (1 - Y)Q]^+, \end{aligned} \quad (26)$$

where  $(x - y)^+$  represents  $\max[(x - y), 0]$ .

White (1970) considers the problem of deciding the optimum batch production quantity when the probability of producing a good-for-sale item is  $p$ , so the total number of good items is distributed Binomial( $Q, p$ ). He shows the expected profit function is strictly concave and derives the optimum batch production quantity. In the following Chapter, we address the concavity of the retailer's expected profit function and derive the optimal order quantity that maximizes the expected profit. We also explore the behavior of the optimal solution when parameters of the logistics damage distribution change.

## CHAPTER 8

### RISK NEUTRAL SOLUTIONS

In this Chapter, we show how the standard expected value approach fails in the case of a low-probability-high-consequence contingency event. Below, we derive the optimal order quantity as a function of model parameters. From equation (26), the retailer's expected profit can be expressed as

$$\begin{aligned}
 E[\Pi(Q, Y)] &= rE[\text{Min}(\xi, (1 - Y)Q)] - c(1 - E[Y])Q \\
 &\quad - hE[((1 - Y)Q - \xi)^+] - \pi E[(\xi - (1 - Y)Q)^+] \\
 &\equiv ESR - ETIC,
 \end{aligned}$$

where  $ESR$  (Expected Sales Revenue) and  $ETIC$  (Expected Total Inventory Cost) are

$$\begin{aligned}
 ESR &= rE[\text{Min}(\xi, (1 - Y)Q)] - c(1 - E[Y])Q \\
 &= rE[\xi] - c(1 - E[Y])Q - r \int_0^1 \int_{(1-y)Q}^\infty [\xi - (1 - y)Q] f(\xi) g(y) d\xi dy, \\
 ETIC &= h \int_0^1 \int_0^{(1-y)Q} [(1 - y)Q - \xi] f(\xi) g(y) d\xi dy \\
 &\quad + \pi \int_0^1 \int_{(1-y)Q}^\infty [\xi - (1 - y)Q] f(\xi) g(y) d\xi dy.
 \end{aligned}$$

Shih (1980) proved the convexity of  $ETIC$ ; to prove the concavity of  $E[\Pi(Q, Y)]$  it suffices to show the concavity of  $ESR$ , which is guaranteed because its second derivative is

$$\frac{\partial^2 ESR}{\partial Q^2} = -r \int_0^1 (1 - y)^2 f((1 - y)Q) g(y) dy < 0, \quad \forall \quad Q. \quad (27)$$

The proof of this result is listed in the Appendix.

For illustration, we consider the simple case in which demand is uniformly distributed with parameters  $a$  and  $b$ :

$$f(\xi) = \begin{cases} (b-a)^{-1} & \text{if } a \leq \xi \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

Under uniformly distributed demand, we have:

$$\int_0^{(1-y)Q} f(\xi) d\xi = \frac{(1-y)Q - a}{b-a}, \quad \int_0^{(1-y)Q} \xi f(\xi) d\xi = \frac{(1-y)^2 Q^2}{2(b-a)}$$

so that expected profit simplifies to:

$$\begin{aligned} E[\Pi(Q, Y)] &= r \frac{a+b}{2} - c(1-\mu)Q - \frac{1}{2(b-a)} \left[ (h+r+\pi)(\sigma^2 + (1-\mu)^2)Q^2 \right. \\ &\quad \left. - 2(1-\mu)(ah + b(r+\pi))Q + a^2h + b^2(r+\pi) \right] \\ &= -\frac{(h+r+\pi)(\sigma^2 + (1-\mu)^2)}{2(b-a)} \left( Q - \frac{(1-\mu)}{(1-\mu)^2 + \sigma^2} \left[ \frac{b(r+\pi-c) + a(h+c)}{r+\pi+h} \right] \right)^2 \\ &\quad + \frac{(a+b)}{2} + a^2h + b^2(r+\pi) + \frac{(1-\mu)^2}{(1-\mu)^2 + \sigma^2} \frac{(b(r+\pi-c) + a(h+c))^2}{2(b-a)(h+r+\pi)} \quad (28) \end{aligned}$$

where  $\mu = E[Y]$ , and  $\sigma^2 = Var[Y]$ . The optimal order quantity  $Q^*$  represents the boundary value between where an increased order provides cost or benefit.

**Proposition 8.0.1** *Let  $Q_0 = \{b(r+\pi-c) + a(h+c)\}/\{r+\pi+h\}$ , which is the optimal order quantity in the conventional news-vendor problem assuming no damages in the order process (i.e.  $\mu = \sigma = 0$ ). In terms of  $Q_0$ , the order quantity which maximizes  $E[\Pi(Q)]$  is*

$$\begin{aligned} Q^* &= \frac{(1-\mu)}{(1-\mu)^2 + \sigma^2} \left( \frac{b(r+\pi-c) + a(h+c)}{r+\pi+h} \right) \\ &= \frac{(1-\mu)}{(1-\mu)^2 + \sigma^2} \times Q_0. \end{aligned}$$

Proposition 8.0.1 shows that the optimal order quantity which maximizes the retailer's expected profits depends only on the mean and the standard deviation of  $Y$ . Then, the

amount received has the form of  $Z = (1 - Y)Q$ , with  $E[Z] = (1 - \mu)Q$  and  $Var[Z] = \sigma^2 Q^2$ .

This result coincides with those results of Noori and Keller (1984) where  $Q^*$  is proportional to  $\mu$  and is reduced by an increase in the variability of  $Y$ .

**Proposition 8.0.2** *The order quantities which maximize ESR and ETIC are, respectively,*

$$Q_A^* = \frac{(1 - \mu)}{(1 - \mu)^2 + \sigma^2} \left( b - (b - a) \frac{c}{r} \right),$$

$$Q_B^* = \frac{(1 - \mu)}{(1 - \mu)^2 + \sigma^2} \left( a + (b - a) \frac{\pi}{\pi + h} \right).$$

Furthermore,  $Q^*$  is a convex combination of  $Q_A^*$  and  $Q_B^*$ :

$$Q^* = \lambda Q_A^* + (1 - \lambda) Q_B^*, \quad \text{where} \quad \lambda = \frac{r}{r + \pi + h}.$$

Proposition 8.0.2 shows that the optimal order quantity is a weighted average of the separate order quantities that maximize ESR ( $Q_A^*$ ) and ETIC ( $Q_B^*$ ); if  $r > \pi + h$ , more weight is assigned to the quantity which maximizes *ESR*. Because the optimal order quantity depends only on the mean and the variance of  $Y$ , we focus on those parameters. If contingency probability is small (e.g.,  $p \leq 0.001$ ), equations (22) through (25) show that

$$\begin{aligned} E[Y_{NM}] &= E[Y_M] \cong \mu_N(2 - \mu_N) \\ Var[Y_{NM}] &\cong \frac{1}{k} \sigma_N^2 [2(1 - \mu_N) + \sigma_N^2] \\ Var[Y_M] &\cong Var[Y_{NM}] + \frac{k-1}{2k} \sigma_N^2 (1 - \mu_N)^2. \end{aligned} \tag{29}$$

The equations in (29) suggest that expected profit does not significantly change under contingency if  $p$  is small enough. A coherent solution for the decision making process must provide a means of protection against the severe effects of contingency; the expected value approach fails to do this. The following Chapter introduces two procedures that allow the

retailer to generate reasonable solutions reflecting natural risk-averse characteristics toward extreme events that have low probability.

## CHAPTER 9

### RISK AVERSE SOLUTIONS

This Chapter discusses two alternative solutions to the expected value approach for optimal ordering. The first method limits the solution space to the set of order quantities which guarantees an expected profit level under contingency. The second method features a constraint based on the quantile function of the profit distribution. While both methods restrict the solution space to control the consequence of the contingency, they differ in important ways; the first method considers only the measured contingency and not its probability while the second method is based directly on the contingency distribution. The following result (see Appendix for proof) is useful to understand the behavior of the retailer's profit function.

**Proposition 9.0.3** *Whenever  $(c + h) > (r + \pi - c)$ , the variability of retailer's profit is increasing in order quantity  $Q$ .*

Proposition 9.0.3 states that whenever the profit margin loss from the unit surplus is greater than that from the unit short, the variance of retailer's profit is an increasing function of  $Q$ .

#### ***9.1 Constrained Optimization I - Profit constraint***

Given a contingent event, we consider only solutions that lead to (conditionally) expected profit of at least  $\Pi_0$ . Let  $I$  denote the indicator function for a contingency. The problem

becomes

$$\begin{aligned} \max_{Q \geq 0} \quad & E_G[\Pi(Q, Y)] \\ \text{s.t.} \quad & E_{G_C}[\Pi(Q, Y)] \equiv E_G[\Pi(Q, Y)|I = 1] \geq \Pi_0. \end{aligned} \quad (30)$$

Retailer's strong risk-aversion can be reflected by increasing the value of  $\Pi_0$ . The restricted solution space is based on the following sets of order quantities:

$$\begin{aligned} S_{\Pi_0} &= \text{Set of possible order quantities which lead to unconditional expected profit } \geq \Pi_0, \\ &= \{Q \mid E_G[\Pi(Q, Y)] \geq \Pi_0\} \\ S_{\Pi_0, C} &= \text{Set of possible order quantities which lead to contingency expected profit } \geq \Pi_0, \\ &= \{Q \mid E_{G_C}[\Pi(Q, Y)] \geq \Pi_0\} \\ \bar{S}_{\Pi_0} &= S_{\Pi_0} \cap S_{\Pi_0, C} \end{aligned}$$

With uniformly distributed demand, the expected profit function in (28) simplifies:

$$E[\Pi(Q, Y)] = -A(\mu, \sigma^2)[Q - B(\mu, \sigma^2)]^2 + C(\mu, \sigma^2), \quad (31)$$

where

$$\begin{aligned} A(\mu, \sigma^2) &= \frac{(h+r+\pi)(\sigma^2+(1-\mu)^2)}{2(b-a)} \text{ determines the spread of the profit function,} \\ B(\mu, \sigma^2) &= \frac{(1-\mu)}{(1-\mu)^2+\sigma^2} \left[ \frac{b(r+\pi-c)+a(h+c)}{r+\pi+h} \right] \text{ determines the optimal order quantity, and} \\ C(\mu, \sigma^2) &= \frac{(a+b)}{2} + a^2h + b^2(r+\pi) + \frac{(1-\mu)^2}{(1-\mu)^2+\sigma^2} \left[ \frac{(b(r+\pi-c)+a(h+c))^2}{2(b-a)(h+r+\pi)} \right] \\ &\text{determines the maximum expected profit.} \end{aligned}$$

From the expression in (31), it is clearly seen that the expected profit only depends on the the mean and the variance of  $Y$ . By partitioning  $E[\Pi(Q, Y)]$  into three parts, its behavior is more clearly manifest when the mean and the variance of the damage distribution vary. Specifically,  $A$  and  $C$  are decreasing functions of  $\mu$  while  $B$  is an increasing function

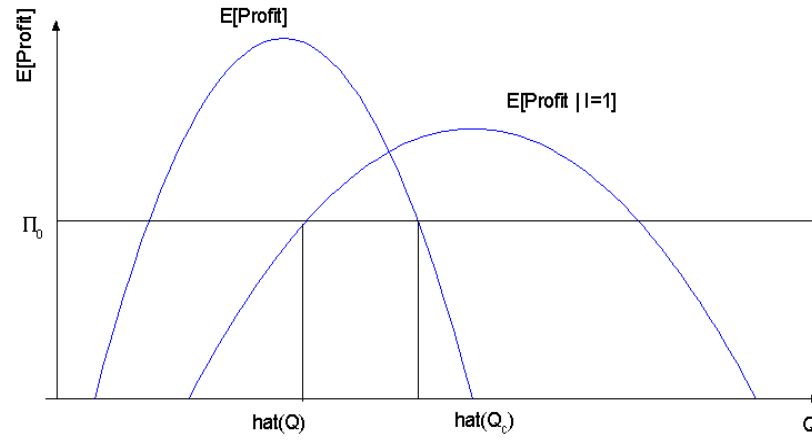
of  $\mu$ .  $A$  is increasing in  $\sigma^2$ , while  $B$  and  $C$  decrease in  $\sigma^2$ . When the mean increases, the expected profit curve broadens ( $\frac{\partial A}{\partial \mu} < 0$ ), shifts to the right (e.g. the optimal order quantity increases) ( $\frac{\partial B}{\partial \mu} > 0$ ), and the corresponding maximum expected profit decreases ( $\frac{\partial C}{\partial \mu} < 0$ ). When the variance increases, the curve shrinks ( $\frac{\partial A}{\partial \sigma^2} > 0$ ), shifts to the left (e.g. the optimal order quantity decreases) ( $\frac{\partial B}{\partial \sigma^2} < 0$ ), and the maximum expected profit decreases ( $\frac{\partial C}{\partial \sigma^2} < 0$ ).

Based on these properties, we consider two ways contingency affects the expected profit through the distribution of the total damage proportion: (1) contingency increases the mean of  $Y$ , and (2) contingency increases the variance of  $Y$ . Let  $\hat{Q}$  = the optimal ordering quantity of the unconditional problem,  $\hat{Q}_C$  = the optimal ordering quantity under contingency, and  $Q^*$  = the optimal solution to the constrained optimization problem in (30). As  $\Pi_0$  increases, the solution  $Q^*$  increases toward  $\hat{Q}_C$ ; as  $\Pi_0$  decreases, the constraint eventually disappears. Figure 5 illustrates the approach with the conditional and unconditional profit functions. This problem formulation provides flexibility to the decision maker, with  $\Pi_0$  serving as a utility function that shrinks the unconstrained solution towards  $\hat{Q}_C$  in the case of contingency.

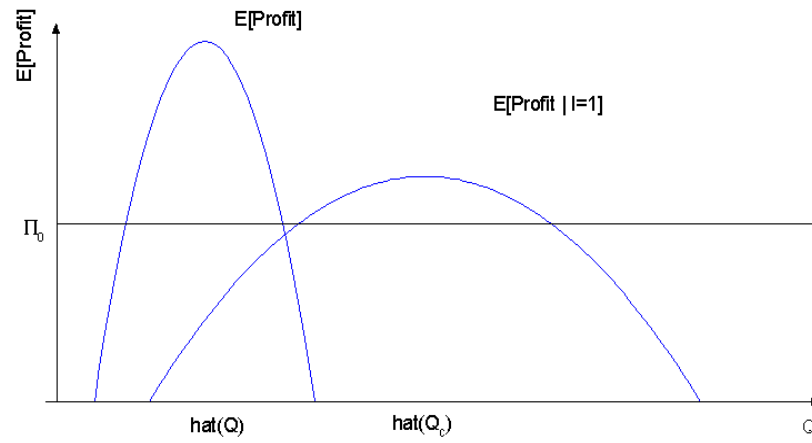
### 9.1.1 Contingency Causes Location Shift in TPDP

We first examine the case where the contingency causes a location shift (to the right) in damage distribution, so  $G_C$  is increased from  $G$  by a constant. The location shift (to the right) in  $G$  causes both the location shift (to the right) and increased variability in the expected profit (see Figure 5). In the case  $\bar{S}_{\Pi_0} = S_{\Pi_0} \cap S_{\Pi_0,C} \neq \emptyset$ , if  $\hat{Q} \in \bar{S}_{\Pi_0}$  then  $Q^* = \hat{Q}$ , otherwise  $Q^* = \min_{Q \in \bar{S}_{\Pi_0}} \{Q\}$ . If  $\bar{S}_{\Pi_0} = \emptyset$ , it is not possible to keep conditional expected profit above  $\Pi_0$  (in case of contingency) without allowing overall expected profit to go below  $\Pi_0$  (see Figure 6). To rectify this problem,  $\Pi_0$  must be reduced until there is an overlap between  $S_{\Pi_0}$  and  $S_{\Pi_0,C}$ . In general, the order quantity increasing in the level of risk-aversion





**Figure 5:** Expected profits based on increased mean in damage distribution after contingency



**Figure 6:** Increased mean causes infeasible solution

(e.g. when  $\Pi_0$  increases,  $Q^*$  increases) under the location shift case.

### 9.1.2 Contingency Causes Increased Variability in TPDP

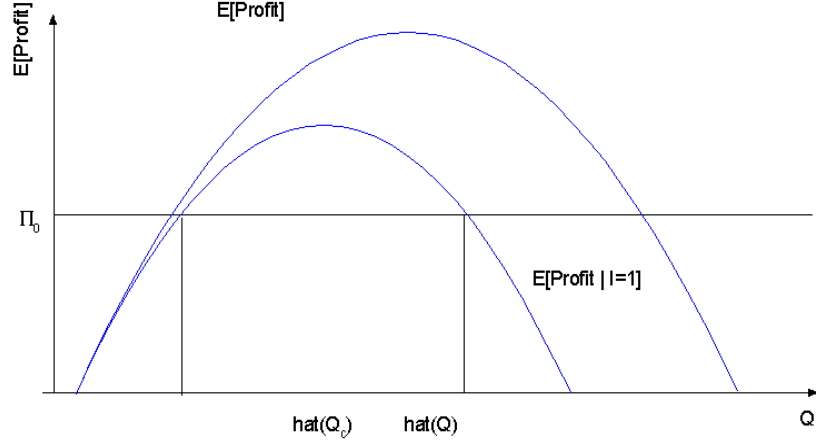
From Proposition 8.0.1, the optimal order quantity which maximizes expected profit decreases as the variance increases. Accordingly, increased variability in  $Y$  shifts the expected profit to the left as illustrated in Figure 7. Furthermore, the increase in variance results in a decrease in expected profit. The vertical distance between the two local maxima in the expected profit curves represents the decrease in maximum possible profits due to the increase in variance. When the variance under contingency is  $\sigma^2 + \delta$ , this distance is

$$\begin{aligned}\Delta &= C(\mu, \sigma^2) - C(\mu, \sigma^2 + \delta) \\ &= \left[ \frac{(1 - \mu)^2}{(1 - \mu)^2 + \sigma^2} - \frac{(1 - \mu)^2}{(1 - \mu)^2 + \sigma^2 + \delta} \right] \left[ \frac{(b(r + \pi - c) + a(h + c))^2}{2(b - a)(h + r + \pi)} \right].\end{aligned}$$

If  $\bar{S}_{\Pi_0} = S_{\Pi_0} \cap S_{\Pi_0, C} \neq \emptyset$ , then a unique solutions exists; if  $\hat{Q} \in \bar{S}_{\Pi_0}$  then  $Q^* = \hat{Q}$ , otherwise  $Q^* = \max_{Q \in \bar{S}_{\Pi_0}} \{Q\}$ . Again, if no overlap between  $S_{\Pi_0}$  and  $S_{\Pi_0, C}$  exists, the constraint  $\Pi_0$  must be reduced. Not like the location shift case, strong risk-aversion (bigger value of  $\Pi_0$ ) reduces the order quantity due to the increased variance. The section that follows illustrates these solution procedures with a numerical example.

## 9.2 Constrained Optimization II - Probability constraint

As an alternative to controlling the profit function by conditioning on the occurrence of a contingency, here we restrict the solutions space by restricting the probability space of the profit distribution. That is, we bound from below (with  $\gamma$ ) the probability that the profit is less than an amount  $\Pi_1$ . If we assume, for simplicity, that the demand level  $\xi$  is fixed, the problem becomes



**Figure 7:** Increased variance shifts expected profits to the left

$$\begin{aligned} \max_{Q \geq 0} \quad & E_g[\Pi(Q, Y)] \\ \text{s.t.} \quad & P_g(\Pi(Q, Y) \leq \Pi_1) \leq \gamma, \end{aligned} \quad (32)$$

where

$$\Pi(Q, Y) = \begin{cases} r\xi - c(1 - Y)Q - h((1 - Y)Q - \xi), & (1 - Y)Q \geq \xi; \\ r\xi - c(1 - Y)Q - (r + \pi)(\xi - (1 - Y)Q), & (1 - Y)Q \leq \xi. \end{cases}$$

In this case, retailer's strong risk-aversion can be reflected by either increasing the value of  $\Pi_1$  or decreasing the value of  $\gamma$ . Denote by  $\hat{Q}$  the optimal order quantity for the unconstrained maximization problem that satisfies the following expression (See Shih (1980)):

$$\int_0^{1-\xi/\hat{Q}} (1 - y)g(y)dy = \frac{(1 - \mu)(r - c + \pi)}{r + h + \pi}. \quad (33)$$

If we tacitly assume  $Q \geq \xi$ , the probability constraint becomes

$$\begin{aligned}
P[\Pi(Q, Y) \leq \Pi_1] &= P\left(Y \geq 1 - \frac{\pi\xi + \Pi_1}{(r + \pi - c)Q} \middle| Y \geq 1 - \frac{\xi}{Q}\right) P\left(Y \geq 1 - \frac{\xi}{Q}\right) \\
&+ P\left(Y \leq 1 - \frac{r\xi + h\xi - \Pi_1}{(h + c)Q} \middle| Y \leq 1 - \frac{\xi}{Q}\right) P\left(Y \leq 1 - \frac{\xi}{Q}\right).
\end{aligned}$$

The following propositions , with the proof in Appendix, characterize the solution to the stochastic constraint placed on the profit distribution.

**Proposition 9.2.1** *For any fixed target profit level  $\Pi_1$ , there exists a critical order size  $Q_1 = \Pi_1/(r - c)$  such that for any demand*

$$Q \leq Q_1 \rightarrow P[\Pi(Q, Y) \leq \Pi_1] = 1$$

$$Q > Q_1 \rightarrow P[\Pi(Q, Y) \leq \Pi_1] = 1 - G\left[1 - \frac{\pi\xi + \Pi_1}{(r + \pi - c)Q}\right] + G\left[1 - \frac{r\xi + h\xi - \Pi_1}{(h + c)Q}\right].$$

**Proposition 9.2.2** *For any given target profit level  $\Pi_1$  and probability  $\gamma$ , there exists a feasible set for (32) of the form  $S(A, B) = \{Q | Q_L \leq Q \leq Q_U\}$  with*

$$\begin{aligned}
Q_L &= \frac{\pi\xi + \Pi_1}{(r + \pi - c)(1 - G^{-1}(1 - \gamma))}, \\
Q_U &= \left\{ Q : \gamma = 1 - G\left[1 - \frac{\pi\xi + \Pi_1}{(r + \pi - c)Q}\right] + G\left[1 - \frac{r\xi + h\xi - \Pi_1}{(h + c)Q}\right] \right\},
\end{aligned}$$

where  $G^{-1}$  represents the inverse cumulative distribution function of  $Y$ .

As  $\Pi_1$  increases, the feasible set  $S$  shrinks; the lower boundary  $Q_L$  decreases in  $\gamma$  while the upper boundary  $Q_U$  increases so  $S$  widens as  $\gamma$  increases.

**Proposition 9.2.3** *If  $\gamma < \gamma_1 = 1 - G(1 - \zeta(\Pi_1))$ , where  $\zeta(t) = \left(\frac{\pi\xi + t}{r + \pi - c}\right)\left(\frac{r\xi + h\xi - t}{h + c}\right)^{-1}$ , then  $S(\Pi_1, \gamma) = \emptyset$ , e.g. there is no feasible solution in (32).*

*Proof* We need to show that  $P[\Pi(Q, Y) \leq \Pi_1]$  has only one minimum point at  $(r\xi + h\xi - \Pi_1)/(h + c)$ . Because  $1 - G\left(1 - (\pi\xi + \Pi_1)/((r + \pi - c)Q)\right)$  is a decreasing function of  $Q$  and

$G(1 - (r\xi + h\xi - \Pi_1)/((h + c)Q_U))$  is increasing in  $Q$ ,  $P[\Pi(Q, Y) \leq \Pi_1]$  has its minimum at  $((r\xi + h\xi - \Pi_1)/(h + c))$ .  $\square$

**Proposition 9.2.4** *If  $Q \geq Q_1$ , then  $\gamma_1$  is increasing in  $\Pi_1$  and  $\gamma_1 = 1$  when  $\Pi_1$  is set at the maximum profit level, e.g.  $\Pi_1 = (r - c)\xi$ , which can be achieved only when there are no product shortages nor unsold products.*

*Proof* It is easy to see that  $\zeta(t)$  is an increasing function of  $t$ . If  $Q \geq Q_1$ , it can be shown that  $\zeta(\Pi_1) \leq 1$ . When  $\Pi_1 = (r - c)\xi$ , then  $\zeta(\Pi_1) = 1$ , hence  $\gamma_1 = 1$ .  $\square$

**Proposition 9.2.5** *For any given profit level  $\Pi_1$  and probability  $\gamma$ ,  $\gamma \geq \gamma_1$ , the optimal order quantity for the constrained optimization problem (32) is*

$$Q^* = \begin{cases} \hat{Q}, & \text{if } \hat{Q} \in S(\Pi_1, \gamma), \\ Q_L, & \text{if } \hat{Q} \leq Q_L, \\ Q_U, & \text{if } \hat{Q} \geq Q_U \end{cases}$$

where  $\hat{Q}$  is determined by (33).

In the previous section, only the mean and the variance of  $Y$  are used to derive the optimal solution, but in this case, the distribution of  $Y$  must be known to apply probability constraints and derive an optimal solution. If the distribution is known along with appropriate values of  $\Pi_1$  and  $\gamma$ , profit loss can be avoided in the case of contingency. However, because of the small probability of contingency, the value of  $\gamma$  must be selected carefully.

### 9.3 Remarks

It is interesting to note that traditional Mean-Variance and Max-Min procedures are not appropriate to generate risk-averse solutions under possible contingency. First, we consider

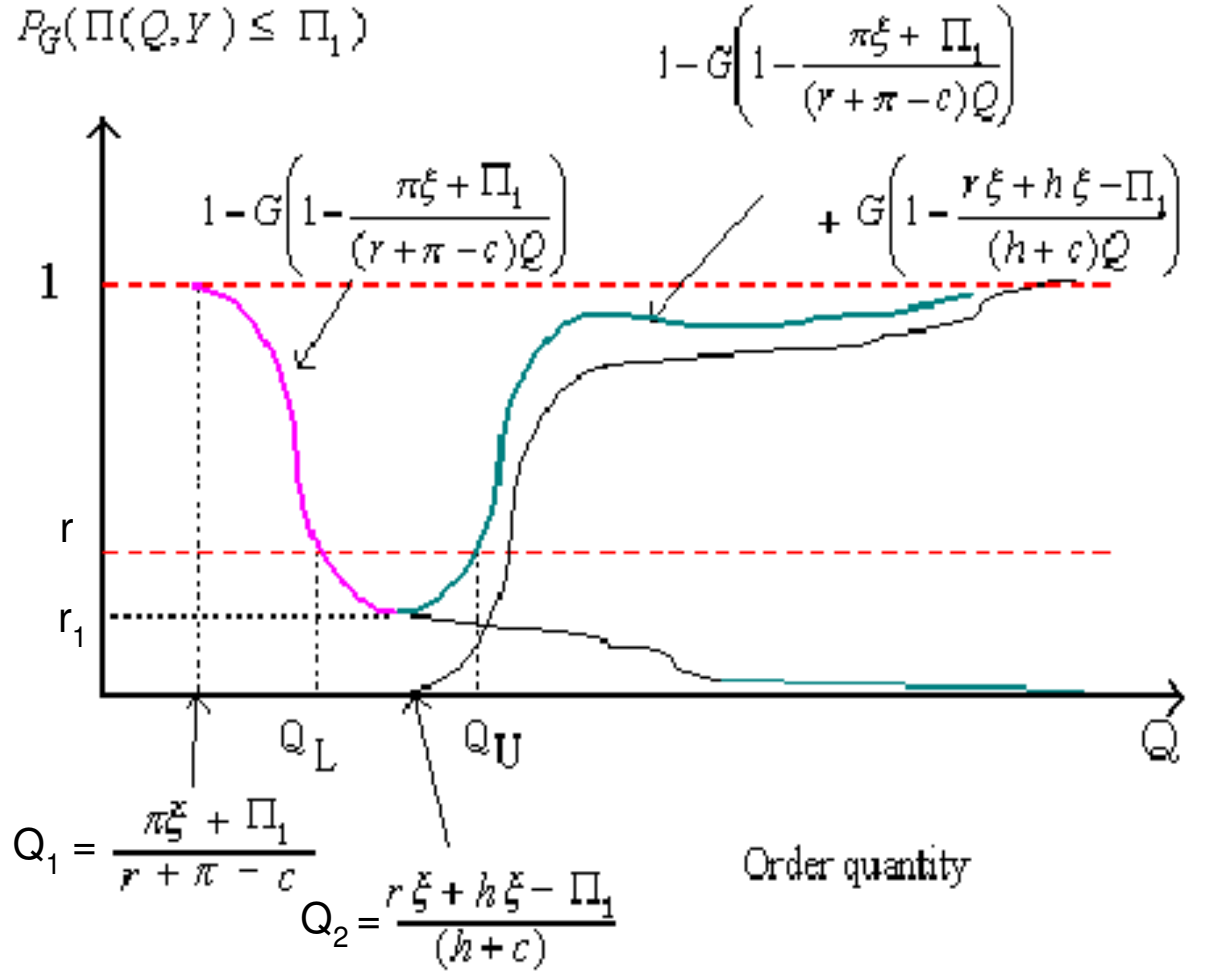


Figure 8: Shape of the probability constraint

the Mean-Variance procedure. The objective function can be written as:

$$\max_{Q \geq 0} E_G[\Pi(Q, Y)] - \alpha Var_G[\Pi(Q, Y)].$$

Then it is not difficult to rewrite it as:

$$\begin{aligned} \max_{Q \geq 0} & (E_{G_N}[\Pi(Q, Y)] - \alpha Var_{G_N}[\Pi(Q, Y)])P[I = 0] \\ & + (E_{G_C}[\Pi(Q, Y)] - \alpha Var_{G_C}[\Pi(Q, Y)])P[I = 1]. \end{aligned}$$

Assuming  $p = P[I = 1]$  is vary small, the solution to the above maximization problem will maximize the objective function under none contingency case. In this way, the contingency cannot come into play to derive risk-averse solutions because of its small probability.

Now, the formal objective function under Max-Min procedure is:

$$\max_Q \min_G E_G[\Pi(Q, Y)]$$

Max-Min procedure provides a solution which maximizes the expected profit under the worst case scenario. Figures in Section (9.1) can be used to illustrate Max-Min solutions. Max-Min solutions of Figure 5 and Figure 6 correspond to ordering quantities where two curves intersect. In Figure 7, the Max-Min solution coincides with the ordering quantity which maximizes the expected profit under contingency. In this way, the Max-Min solution does not provide any flexibility in terms of the resulting ordering quantity. Contrary to these two traditional methods for risk-averse solutions, the two proposed methods in this section provide not only a way to handle the low probability events but also a great flexibility in deriving ordering quantity decisions in accordance with decisionmaker's various degrees of risk preferences by adjusting parameter values.

**Table 5:** Parameter values in the case of contingency

$(\mu_c, \sigma_c^2)$	(0.05, 0.01)	(0.1, 0.01)	(0.2, 0.01)	(0.3, 0.01)
	(0.4, 0.01)	(0.5, 0.01)	(0.6, 0.01)	(0.7, 0.01)
$(\mu, \sigma_c^2)$	(0.01, 0.05)	(0.01, 0.1)	(0.01, 0.2)	(0.01, 0.3)
	(0.01, 0.4)	(0.01, 0.5)	(0.01, 0.6)	(0.01, 0.7)

**Table 6:** Example: case 1 vs. case 2

Mean	Variance	$\hat{Q}$	E(Profit)	%	Mean	Variance	$\hat{Q}$	E(Profit)	%
0.01	0.01	143	\$4,575	100.0	0.01	0.01	143	\$4,575	100.0
0.05	0.01	149	\$4,561	0.3	0.01	0.05	137	\$3,934	14.0
0.1	0.01	157	\$4,540	0.8	0.01	0.1	131	\$3,198	30.1
0.2	0.01	176	\$4,487	1.9	0.01	0.2	120	\$1,915	58.1
0.3	0.01	200	\$4,410	3.6	0.01	0.3	110	\$831	81.8
0.4	0.01	231	\$4,293	6.2	0.01	0.4	102	-\$95	102.1
0.5	0.01	274	\$4,102	10.3	0.01	0.5	95	-\$896	119.6
0.6	0.01	336	\$3,762	17.8	0.01	0.6	89	-\$1,595	134.9
0.7	0.01	428	\$3,075	32.8	0.01	0.7	84	-\$2,211	148.3

## 9.4 Numerical Examples

### 9.4.1 Constrained Optimization I - Profit constraint

For the following example, we assign parameter values:

$$(r, c, \pi, h) = (\$50, \$10, \$30, \$2)$$

with the demand distribution of  $U(a = 100, b = 150)$ . We assume the (unconditional) mean and the variance of  $Y$  are  $(\mu, \sigma^2) = (0.01, 0.01)$ . In the case of contingency, we use the parameter values listed in Table 5. We let  $\mu_c$  and  $\sigma_c^2$  represent the mean and the variance of  $Y$  under contingency, respectively.

Table 6 lists optimal ordering quantities which maximize expected profits under various combinations of mean and variance of  $Y$ . As stated previously,  $\hat{Q}$  increases as the mean of  $Y$  increases and decreases as the variance of  $Y$  increases. For the unconditional case, the optimal ordering quantity  $\hat{Q} = 143$  with  $E[\Pi(\hat{Q})] = \$4,575$ . If the expected profit under



**Table 7:** Solutions when  $\Pi_0 = \$4,000$  and  $\sigma^2 = 0.01$ 

$\mu_c$	$S_{4000,C}$	$S_{4000}$	$\bar{S}_{4000}$	$Q^*$	$E[\Pi(Q^*)]$
0.05	122, 175	117, 169	122, 169	143	4575
0.1	129, 184	117, 169	129, 169	143	4575
0.2	146, 205	117, 169	146, 169	146	4566
0.3	169, 231	117, 169	169, 169	169	4012
0.4	201, 262	117, 169	$\emptyset$	201	1813
0.5	253, 296	117, 169	$\emptyset$	253	-5308
0.6	$\emptyset$	117, 169	$\emptyset$	infeasible	NA
0.7	$\emptyset$	117, 169	$\emptyset$	infeasible	NA

contingency is restricted to be at or above  $\Pi_0 = \$4,000$ , the solution satisfies

$$\begin{aligned} \max_{Q \geq 0} \quad & E_g[\Pi(Q, Y)] \\ \text{s.t.} \quad & E_{g_C}[\Pi(Q, Y)] \geq 4000. \end{aligned}$$

When contingency increases the mean to  $\mu_c = 0.05$ , we have  $\bar{S}_{4000} = S_{4000} \cap S_{4000,C} = \{117 \leq Q \leq 169\} \cap \{122 \leq Q \leq 175\} = \{122 \leq Q \leq 169\}$ . The solution  $Q^* = \hat{Q} = 143$  because  $143 \in \bar{S}_{4000}$ . Table 7 summarizes results using different values of  $\mu_c$ . When  $\mu_c \geq 0.4$ ,  $\bar{S}_{4000}$  is the empty set, so there is no solution which guarantees a minimum expected profit of \$4,000 regardless of contingency. If we choose a constraint value of  $\Pi_0$  smaller than \$4,000, say \$3,000, we have nonempty set of  $\bar{S}_{3000}$  when  $\mu_c = 0.4$  as shown in Table 8. Figure 9 shows plots of expected profits versus ordering quantity at different levels of  $\mu$  given  $\sigma^2 = 0.01$ .

If contingency leads to the increase in variance of  $Y$ , the optimal ordering quantity decreases. For example, at  $\Pi_0 = \$3,000$ , Table 9 shows solutions for eight different values of  $\sigma_c^2$  between 0.05 and 0.70. In Figure 8, we fix the mean at a constant value ( $\mu = 0.01$ ), and change the variance to see the effect on expected profit. As expected, the optimal order

**Table 8:** Solutions when  $\Pi_0 = \$3,000$  and  $\sigma^2 = 0.01$ 

$\mu_c$	$S_{3000,C}$	$S_{3000}$	$\tilde{S}_{3000}$	$Q^*$	$E[\Pi(Q^*)]$
0.05	103, 194	99, 186	103, 186	143	4575
0.1	109, 204	99, 186	109, 186	143	4575
0.2	123, 228	99, 186	123, 186	143	4575
0.3	142, 258	99, 186	142, 186	143	4575
0.4	167, 296	99, 186	167, 186	167	4095
0.5	203, 346	99, 186	$\emptyset$	203	1620
0.6	262, 409	99, 186	$\emptyset$	262	-6986
0.7	398, 411	99, 186	$\emptyset$	398	-48355

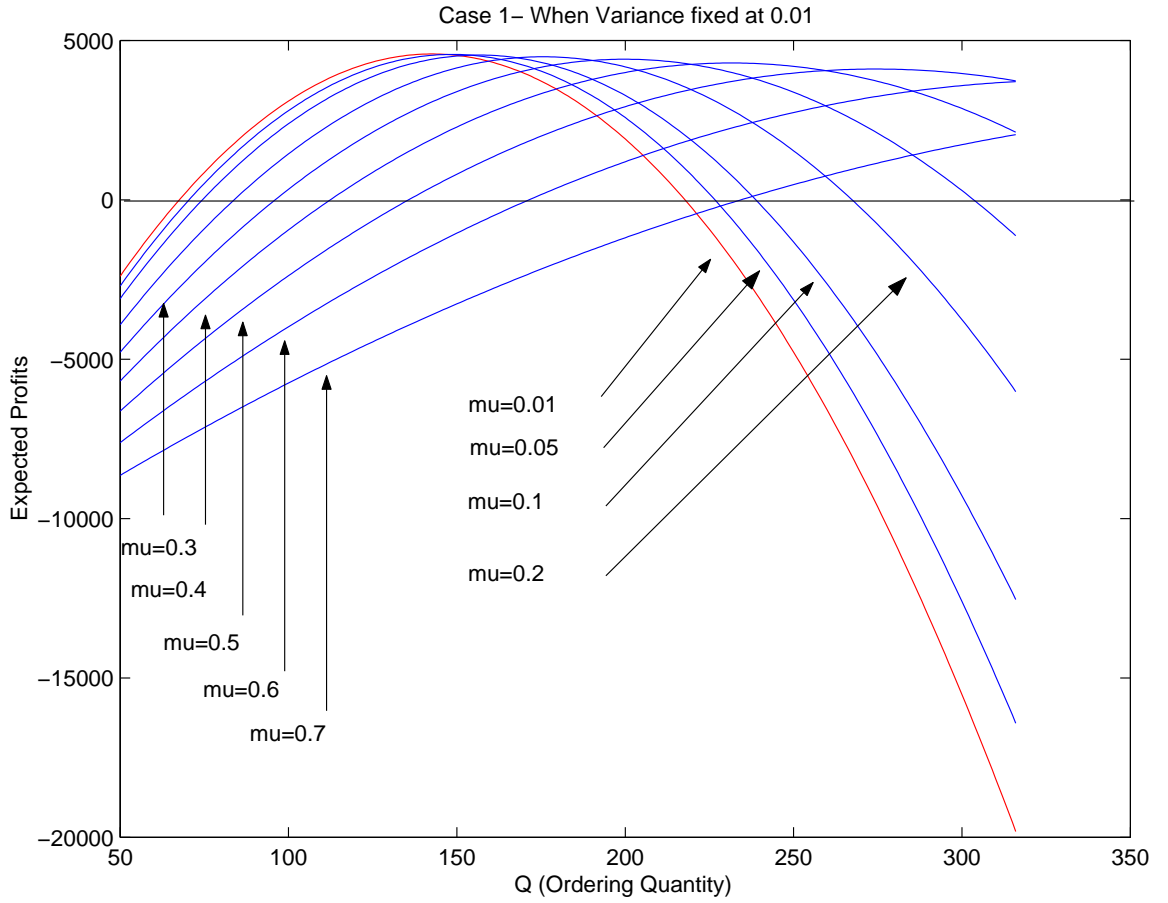
quantity increases as the mean of  $Y$  increases and decreases as the variance increases. It can be seen that the maximum expected profit is much more sensitive to changes in the variance compared to changes in the mean. The expected profit loss from the increase in the mean can be compensated by increasing the quantity of the order. The profit loss from an increase in variance can be much more dramatic, however, and the constrained maximization problem becomes infeasible quickly in this case. Profit loss becomes worse with larger order quantities because the larger order naturally creates more variability in the amount of total damage. The proposed approach provides a more robust solution to the optimization problem and is more appropriate when the contingency increases the mean of  $Y$  while variance remains stable. Otherwise, even with a reasonable constraint amount  $\Pi_0$ , the feasible set  $S_{\Pi_0}$  can be empty.

#### 9.4.2 Constrained Optimization II - Probability constraint

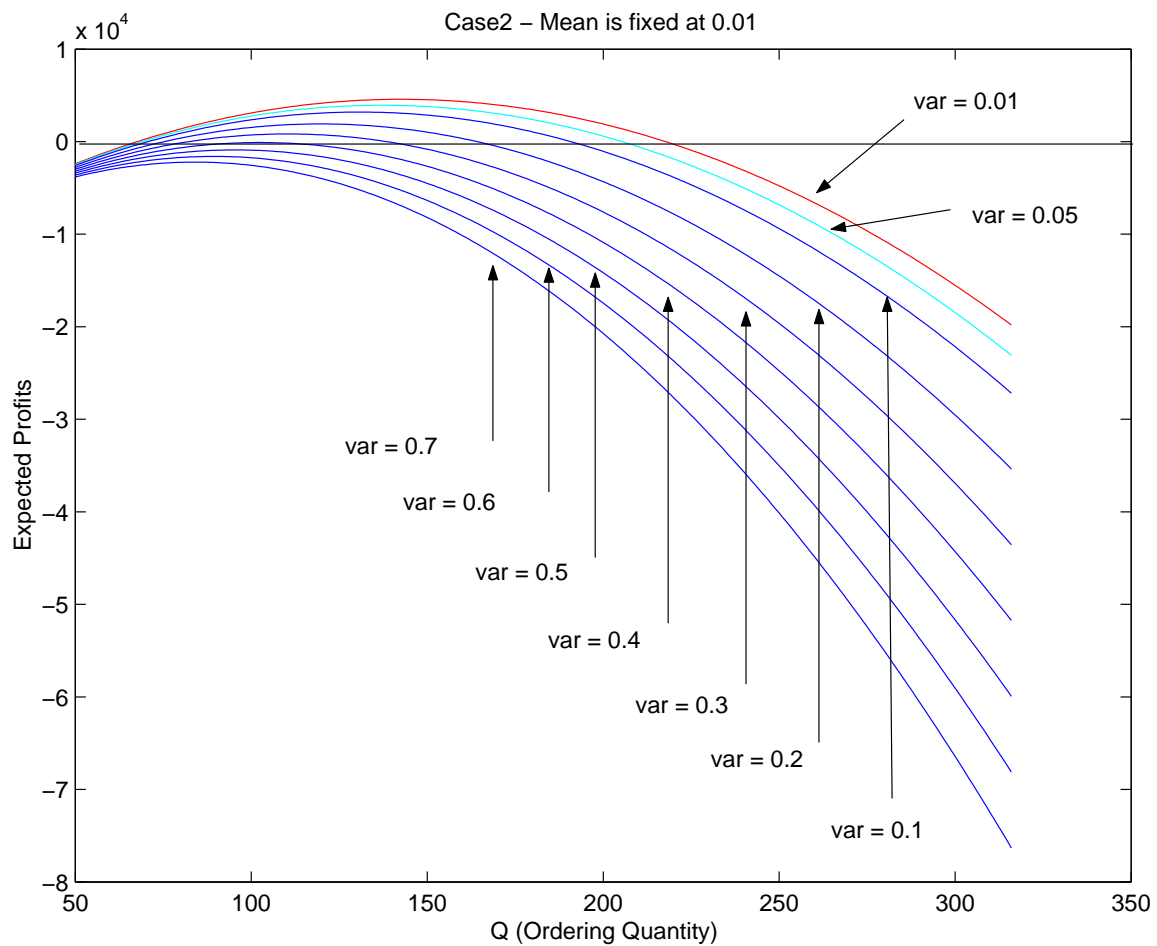
This example uses the same parameter values as before,  $(r, \pi, h) = (\$50, \$30, \$2)$ , with a fixed demand level at 120 units per period and  $k = 2$  suppliers, and the following distributional assumptions:

**Table 9:** Solutions when  $\Pi_0=\$3,000$  and  $\mu = 0.01$

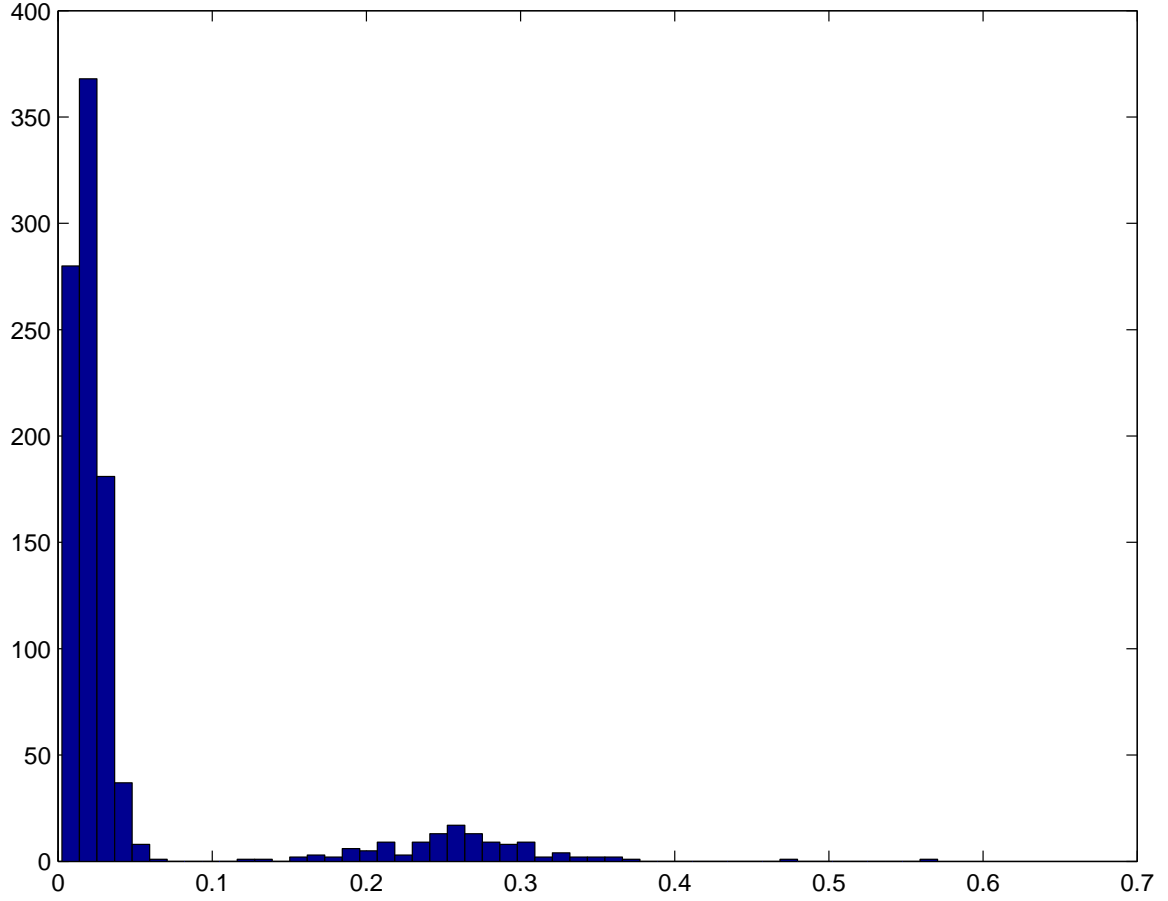
$\sigma_c^2$	$S_{3000,C}$	$S_{3000}$	$\bar{S}_{3000}$	$Q^*$	$E[\Pi(Q^*)]$
0.05	104, 170	99, 186	104, 170	143	4575
0.1	116, 145	99, 186	116, 145	143	4575
0.2	$\emptyset$	99, 186	$\emptyset$	infeasible	NA
0.3	$\emptyset$	99, 186	$\emptyset$	infeasible	NA
0.4	$\emptyset$	99, 186	$\emptyset$	infeasible	NA
0.5	$\emptyset$	99, 186	$\emptyset$	infeasible	NA
0.6	$\emptyset$	99, 186	$\emptyset$	infeasible	NA
0.7	$\emptyset$	99, 186	$\emptyset$	infeasible	NA



**Figure 9:** Profit functions under fixed variance



**Figure 10:** Profit functions under fixed mean



**Figure 11:** Simulated *pdf* of  $Y_{NM}$

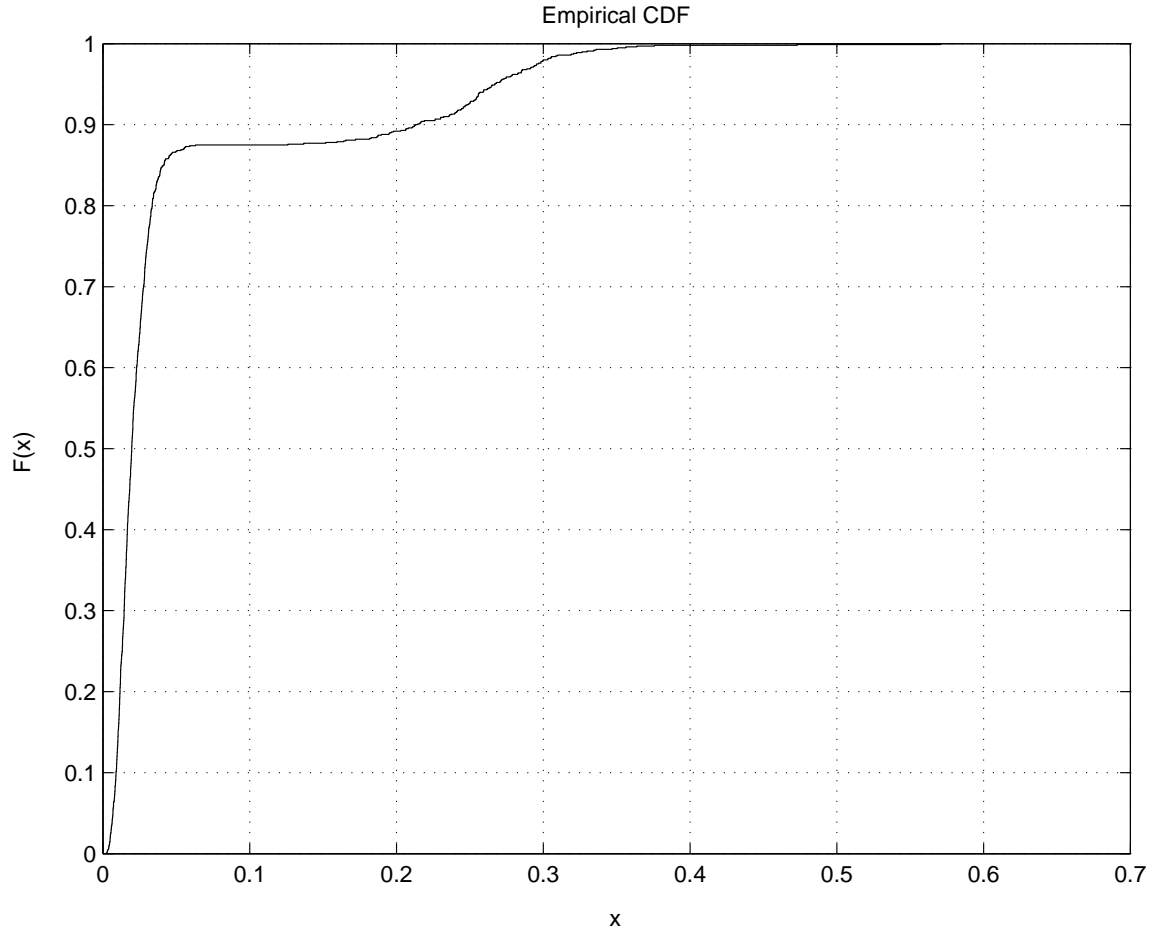
$$X_{jC}, X^*_{jC} \sim iid \text{Beta}(10, 10)$$

$$X_{jN}, X^*_{jN} \sim iid \text{Beta}(1, 99)$$

$$I_j, I^*_j, I_0 \sim iid \text{Bernoulli with } p = 0.01.$$

Figures 11 and 12 show the shape of the simulated distribution of  $Y$  under the non-mixing assumption. Four different values of  $\Pi_1$  and three different probabilities for  $\gamma$  are considered:  $\{\Pi_1 \in 1000, 2000, 3000, 4000\}$  and  $\{\gamma \in 0.1, 0.01, 0.001\}$ . Table 10 summarizes values of considered parameters.

The optimal order quantity ( $\hat{Q}$ ) for the unconstrained maximization problem is 124 units independent of two different logistics operations. Resulting solutions under “Separated” and “Integrated” logistics operations (described in Section 3) are summarized in



**Figure 12:** Empirical *cdf* of  $Y_{NM}$

**Table 10:** Parameter values

Parameter	Value
$r$	\$50
$\xi$	120 units per period
$h$	\$2
$\pi$	\$30
$\gamma$	0.1, 0.01, 0.001
$\Pi_1$	\$3000, \$4000
$(c_1, c_2)$	(\$10, \$10), (\$5, \$15), (\$1, \$19)
$X_{jC}, X_C^*$	i.i.d. Beta(10, 10)
$X_{jN}, X_N^*$	i.i.d. Beta(1, 99)
$I_j, I_j^*, I_0$	i.i.d. Bernoulli with $p=0.01$

<b>c1=c2=10</b>		<b>Separated</b>		<b>Integrated</b>	
	r	Q*	E[Q*]	Q*	E[Q*]
3000	0.1	124	\$4,703	124	\$4,690
3000	0.01	132	\$4,661	134	\$4,630
3000	0.001	139	\$4,598	154	\$4,432
4000	0.1	124	\$4,703	124	\$4,690
4000	0.01	152	\$4,477	154	\$4,432
4000	0.001	160	\$4,398	177	\$4,190

<b>c1=5, c2=15</b>		<b>Separated</b>		<b>Integrated</b>	
	r	Q*	E[Q*]	Q*	E[Q*]
3000	0.1	124	\$4,703	124	\$4,690
3000	0.01	124	\$4,703	130	\$4,645
3000	0.001	129	\$4,663	161	\$4,332
4000	0.1	124	\$4,703	124	\$4,690
4000	0.01	140	\$4,561	150	\$4,445
4000	0.001	149	\$4,476	186	\$4,057

<b>c1=1, c2=19</b>		<b>Separated</b>		<b>Integrated</b>	
	r	Q*	E[Q*]	Q*	E[Q*]
3000	0.1	124	\$4,703	124	\$4,690
3000	0.01	124	\$4,703	124	\$4,690
3000	0.001	137	\$4,589	158	\$4,363
4000	0.1	124	\$4,703	124	\$4,690
4000	0.01	143	\$4,533	142	\$4,525
4000	0.001	158	\$4,387	182	\$4,102

**Figure 13:** Solutions under separated and integrated logistics operations

Figure 13. Furthermore, Figure 14 gives detailed comparisons between solutions from two cases graphically. For each fixed value of  $\Pi_1$ , the optimal order quantity  $Q^*$  increases as  $\gamma$  decreases, reflecting strong risk-aversion of decision maker. In all cases, curves representing expected profit at the resulting solutions under separated logistics operations never move below those curves under integrated logistics operations. From this, we can conclude that when products' retail prices, holding costs, and shortage costs are identical, separated logistics channels for products from different suppliers always produce more profits than the integrated logistics channel, regardless of products's purchase costs.

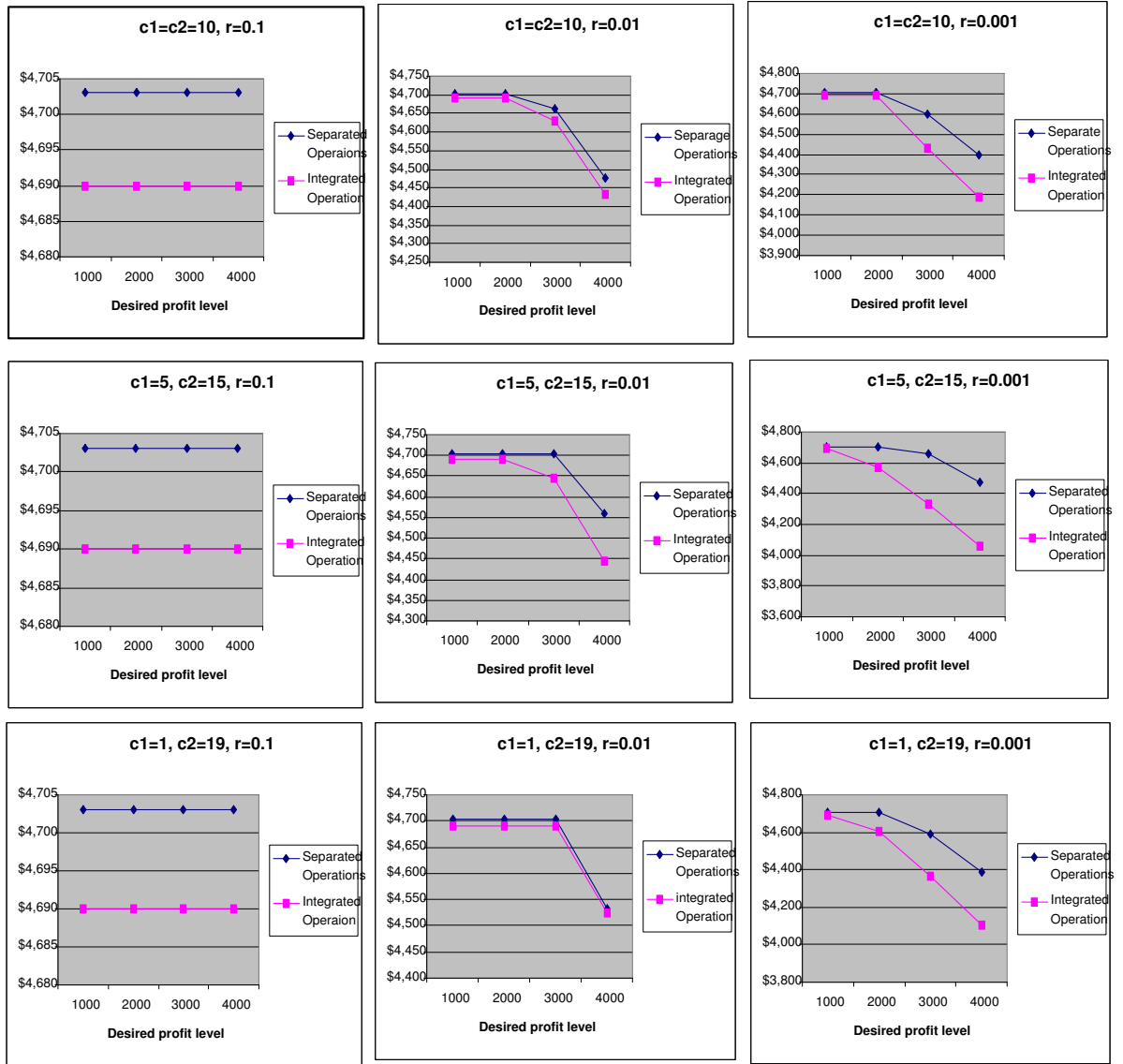


Figure 14: Comparison: Separated vs Integrated



### 9.4.3 Constrained Optimization II - Risk-Pooling using Integrated Operations

In this section, we illustrate that the integrated logistics operation performs better than separated one under certain conditions, such as different products' retail prices, different inventory holding costs, or different products' shortage costs. For illustrations, we introduce different products' shortages costs for our example. We first use simple example to convey the idea and move on to nontrivial example later.

Suppose that proportions of damaged products from two suppliers under separate logistics operations and integrated logistics operation are distributed as following:

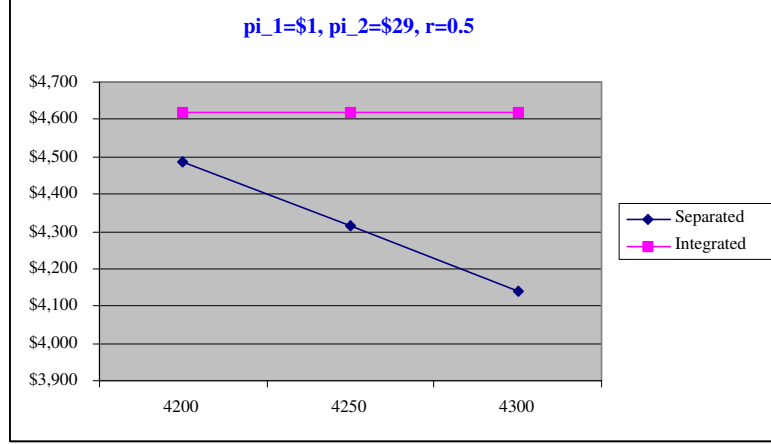
$$P_1(\text{and } P_2) = \begin{cases} 0, & \text{w/p } 0.5; \\ 0.2, & \text{w/p } 0.5. \end{cases} \quad P'_1(\text{and } P'_2) = \begin{cases} 0, & \text{w/p } 0.5; \\ 0.2, & \text{w/p } 0.5. \end{cases}$$

Separated logistics operations (e.g. two trucks to separately ship products from two suppliers) result in independence of  $P_1$  and  $P_2$  while integrated logistics operation makes  $P'_1$  and  $P'_2$  dependent. We adopt the shock model to explain the dependency in case of integrated operation. When only one truck is used for products from two suppliers, it is possible to have that either none of products become damaged or both products from two suppliers become damaged. Based on these, we construct following distributions for total proportion of damaged products:

$$Y_{NM} = \begin{cases} 0, & \text{w/p } 0.25; \\ 0.1, & \text{w/p } 0.5; \\ 0.2, & \text{w/p } 0.25. \end{cases} \quad Y_M = \begin{cases} 0, & \text{w/p } 0.5; \\ 0.2, & \text{w/p } 0.5. \end{cases}$$

We use different shortage costs for products from different suppliers ( $\pi_1 = \$1, \pi_2 = \$29$ ).

$\gamma$  is fixed at 0.5 and  $\Pi_1 = (\$4200, \$4250, \$4300)$  are used. All other parameter values are



**Figure 15:** Risk-pool effects of integrated logistics operation

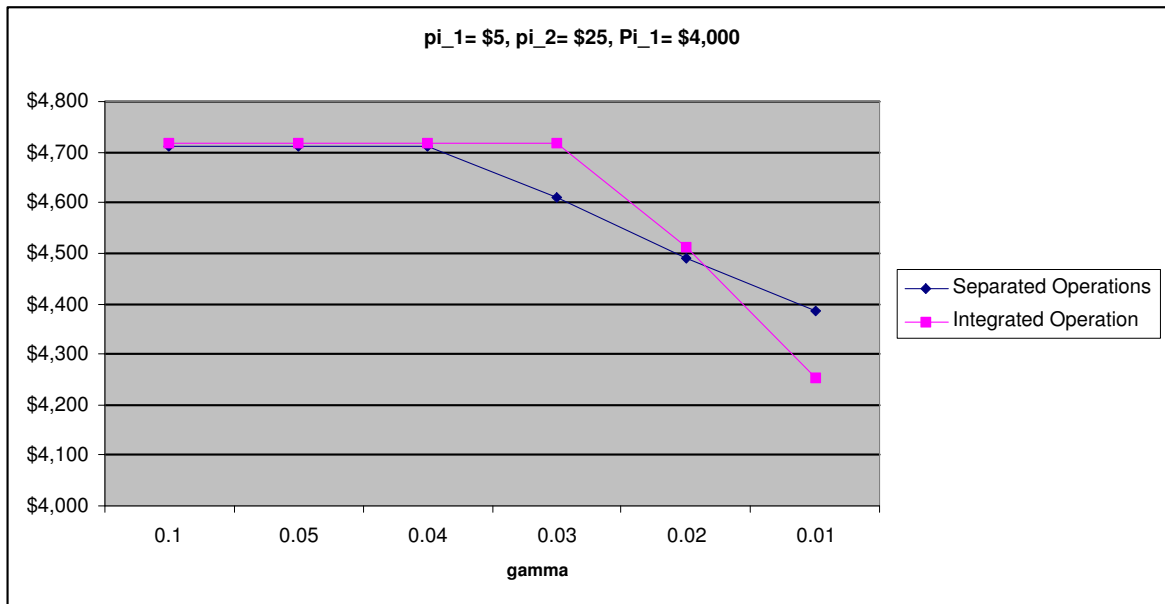
**Table 11:** Distributions Assumptions

$X_{jC}, X_C^*$	i.i.d. Beta(10, 10)
$X_{jN}, X_N^*$	i.i.d. Beta(1, 99)
$I_j, I_j^*, I_0$	i.i.d. Bernoulli with $p=0.01$

kept same as in 9.4.2. Then, it is not difficult to check that integrated logistics operation performs better than separated operations as shown in Figure 15. This result shows the risk-pooling effects of integrated logistics operation. In the following example, we illustrate this risk-pooling effects of integrated operation with more general distributional assumptions. Other than using simple two-point mass discrete distributions as in the first example, we assume following set of distributions as in Table 11.  $c = c_1 = c_2 = \$10$ ,  $\Pi_1 = \$4000$  and  $\gamma = (0.01, 0.02, 0.03, 0.04, 0.05, 0.1)$  are used as well as  $\pi_1 = \$5, \pi_2 = \$25$ . All other parameters are kept same as in Table 10. The results are summarized in Figure 16 and Figure 17. According to these results, it can be said that the integrated logistics operation performs better than separated logistics operations under certain cases (e.g. parameters of  $\Pi_1 = \$4000$  and  $\gamma = (0.02, 0.03, 0.04, 0.05, 0.1)$  in the example).

pi <sub>1</sub> =5, pi <sub>2</sub> =25		separated		integrated	
Pi <sub>1</sub>	gamma	Q*	E[Q*]	Q*	E[Q*]
4000	0.1	124	\$4,712	124	\$4,718
4000	0.05	124	\$4,712	124	\$4,718
4000	0.04	124	\$4,712	124	\$4,718
4000	0.03	135	\$4,610	124	\$4,718
4000	0.02	147	\$4,490	144	\$4,513
4000	0.01	157	\$4,387	168	\$4,254

**Figure 16:** Solutions under separated and integrated logistics operations with different shortage costs



**Figure 17:** Risk-pooling effects of integrated logistics operation

## CHAPTER 10

### CONCLUSION OF PART II

As a brief conclusion, the main goal of this research is to build a bridge between the quantitative uncertain-supply problem and the problem of logistics network planning and vehicle routing, so that the resulting bridge will incorporate supply chain logistics uncertainties in product-ordering decisions. We adopt probability and statistical concepts to better understand the underlying uncertain phenomena. In this thesis, we also investigate the impact of two different logistics operational policies on the resulting solutions.

To achieve our goal, we examine the consequences of supply disruption on the retailer's profits. The supply disruptions take the form of high-impact and low-probability contingencies which can threaten large sections of the supply chain. The traditional expected-value approaches for product-ordering decisions fail to provide the retailer with any means of protection against the effects of contingency. To provide the decision-maker who is facing possible contingency with a systematic way to handle this situation, we propose two procedures in this thesis.

In the first procedure, by constraining the maximization problem with respect to conditional expected profit, a more stable and risk-averse solution can be found. We consider two cases where contingency either increases the mean of the proportion of damage distribution or increases the variance of the distribution. An increase in variance causes a rapid drop in expected profit, leaving no other alternatives to compensate the profit loss. On the other hand, the more robust methods introduced here compensate for a mean shift of the profit

curve, resulting in an increased quantity of the order. In practice, it is recommended that the retailer investigate the characteristic of potential contingency to see how and if it affects the mean or variance of  $Y$ . If the model implies that the contingency changes only the mean, then the retailer can benefit from the constrained optimization solution in Chapter 9. However, if the contingency adversely affects the variance, the decision maker should try to find a way to reduce the variance using, such as, multiple sourcing or purchasing options by which the damage distribution can be truncated.

In the other procedure, we utilize the probability constraint to restrict the resulting solutions. With this procedure, the decision-maker has more options to include his risk preferences in the solution. He can change either the target profit level or the target probability level to derive his risk-averse solutions.

We use the latter procedure to illustrate the risk-pooling effects of the integrated logistics operations under certain conditions. Separated logistics operations between the distribution center and the retailer always generate solutions with higher resulting expected profits compared to those of the integrated logistics operation case whenever the inventory holding cost, the shortage cost, and retail prices of those products from different suppliers are identical. Our examples show that the resulting expected profits may be higher under the integrated logistics operation strategy than the expected profits under separated logistics operations when shortage costs are significantly different.

The investigation of the effects of non i.i.d. defect distributions associated with different routes can be considered for our subsequent research. To make our model more practical, we also need to introduce the logistics cost element in the retailer's profit function. In that case, it will be an interesting problem to study the systematic trade-off methods between the cost saving effects due to the reduced variance from separated logistics operations and the

additional logistics costs required for separated logistics operations. Extending our problem to a multi-period setting is another challenging task.

## CHAPTER 11

### APPENDIX

**Proof of Equation (27):** The first derivative is:

$$\begin{aligned}\frac{\partial ESR}{\partial Q} &= -c(1-\mu) - r \frac{\partial}{\partial Q} \left( \int_0^1 \int_{(1-y)Q}^{\infty} [\xi - (1-y)Q] f(\xi) g(y) d\xi dy \right) \\ &= -c(1-\mu) - r \frac{\partial}{\partial Q} \left( \int_0^1 \int_{(1-y)Q}^{\infty} \xi f(\xi) d\xi g(y) dy - \int_0^1 (1-y)Q \int_{(1-y)Q}^{\infty} f(\xi) d\xi g(y) dy \right)\end{aligned}$$

where

$$\begin{aligned}\frac{\partial}{\partial Q} \left( \int_0^1 \int_{(1-y)Q}^{\infty} \xi f(\xi) d\xi g(y) dy \right) &= - \int_0^1 (1-y)^2 Q f((1-y)Q) g(y) dy, \\ \frac{\partial}{\partial Q} \left( \int_0^1 (1-y)Q \int_{(1-y)Q}^{\infty} f(\xi) d\xi g(y) dy \right) &= \int_0^1 (1-y) \bar{F}((1-y)Q) g(y) dy \\ &\quad - \int_0^1 (1-y)^2 Q f((1-y)Q) g(y) dy.\end{aligned}$$

If we simplify the above expression we have:

$$\frac{\partial ESR}{\partial Q} = -c(1-\mu) + r \int_0^1 (1-y) \bar{F}((1-y)Q) g(y) dy.$$

And now it is easy to see the following result:

$$\frac{\partial^2 ESR}{\partial Q^2} = -r \int_0^1 (1-y)^2 f((1-y)Q) g(y) dy$$

which is negative for all possible values of  $Q$ . □

**Proof of Proposition 9.0.3:** The variance of the profit is

$$\begin{aligned}
Var[\Pi(Q, Y)] &= Var\left(\Pi(Q, Y)|Y \leq 1 - \frac{\xi}{Q}\right)P\left(Y \leq 1 - \frac{\xi}{Q}\right) \\
&\quad + Var\left(\Pi(Q, Y)|Y > 1 - \frac{\xi}{Q}\right)P\left(Y > 1 - \frac{\xi}{Q}\right) \\
&= Var\left(c(1 - Y)Q + h((1 - Y)Q - \xi)\right)P\left(Y \leq 1 - \frac{\xi}{Q}\right) \\
&\quad + Var\left(c(1 - Y)Q + (r + \pi)(\xi - (1 - Y)Q)\right)P\left(Y > 1 - \frac{\xi}{Q}\right) \\
&= (c + h)^2 Var[(1 - Y)Q]G\left(1 - \frac{\xi}{Q}\right) \\
&\quad + (r + \pi - c)^2 Var[(1 - Y)Q]\left(1 - G\left(1 - \frac{\xi}{Q}\right)\right) \\
&= (c + h)^2 Q^2 \sigma^2 G\left(1 - \frac{\xi}{Q}\right) + (r + \pi - c)^2 Q^2 \sigma^2 \left(1 - G\left(1 - \frac{\xi}{Q}\right)\right) \\
&= \left\{ (c + h)^2 G\left(1 - \frac{\xi}{Q}\right) + (r + \pi - c)^2 \left(1 - G\left(1 - \frac{\xi}{Q}\right)\right) \right\} Q^2 \sigma^2
\end{aligned}$$

$$\begin{aligned}
\frac{\partial Var[\Pi(Q, Y)]}{\partial Q} &= 2\sigma^2 Q \left\{ (c + h)^2 G\left(1 - \frac{\xi}{Q}\right) + (r + \pi - c)^2 \left[1 - G\left(1 - \frac{\xi}{Q}\right)\right] \right\} \\
&\quad + \sigma^2 \xi \left( (c + h)^2 - (r + \pi - c)^2 \right) g\left(1 - \frac{\xi}{Q}\right)
\end{aligned}$$

The first term is always positive and the second term is positive whenever  $(c + h) > (r + \pi - c)$ .

□



**Proof of Proposition 9.2.1:** If  $Q \leq Q_1$ , using the previous assumption,  $Q \geq \xi$ , we can show that

$$\frac{\pi\xi + \Pi_1}{(r + \pi - c)} \geq \frac{\pi\xi + (r - c)Q}{(r + \pi - c)} \geq \frac{\pi\xi + (r - c)\xi}{(r + \pi - c)} = \xi,$$

thus,  $\{Y \geq 1 - \frac{\pi\xi + \Pi_1}{(r + \pi - c)Q}\} \cap \{Y \geq 1 - \frac{\xi}{Q}\} = \{Y \geq 1 - \frac{\xi}{Q}\}$  and similarly

$$\frac{r\xi + h\xi - \Pi_1}{(h + c)} \leq \frac{r\xi + h\xi - (r - c)Q}{(h + c)} \leq \frac{r\xi + h\xi - (r - c)\xi}{(h + c)} = \xi,$$

and  $\{Y \leq 1 - \frac{r\xi + h\xi - \Pi_1}{(h + c)Q}\} \cap \{Y \leq 1 - \frac{\xi}{Q}\} = \{Y \leq 1 - \frac{\xi}{Q}\}$ . From these results, the conditional probabilities from the first and the second term in  $P[\Pi(Q, Y) \leq \Pi_1]$  become equal to 1 so that  $P[\Pi(Q, Y) \leq \Pi_1] = P(Y \geq 1 - \frac{\xi}{Q}) + P(Y \leq 1 - \frac{\xi}{Q}) = 1$ . If  $Q > Q_1$ , then  $\{Y \geq 1 - \frac{\pi\xi + \Pi_1}{(r + \pi - c)Q}\} \cap \{Y \geq 1 - \frac{\xi}{Q}\} = \{Y \geq 1 - \frac{\pi\xi + \Pi_1}{(r + \pi - c)Q}\}$  and  $\{Y \leq 1 - \frac{r\xi + h\xi - \Pi_1}{(h + c)Q}\} \cap \{Y \leq 1 - \frac{\xi}{Q}\} = \{Y \leq 1 - \frac{r\xi + h\xi - \Pi_1}{(h + c)Q}\}$  to yield

$$P[\Pi(Q, Y) \leq \Pi_1] = 1 - G\left(1 - \frac{\pi\xi + \Pi_1}{(r + \pi - c)Q}\right) + G\left(1 - \frac{r\xi + h\xi - \Pi_1}{(h + c)Q}\right)$$

□

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## VITA

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